Q.P. 38582

MATHEMATICS SOLUTION

(MAY 2018 SEM 4 MECHANICAL)

Q1) (a) If λ is eigen value of matrix A, then prove that $\lambda^n$is a eigen value of A^n and hence find the eigen values for $A^2 + 2A + 5I$, where $\begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$. (5M)

Solution:

Since λ is an eigenvalue of A if X is the corresponding eigenvector.

AX=λX

Pre-multiply by A,

AAX = λAX

$A^2X = \lambda AX = \lambda\lambda X = \lambda^2 X$

$A^2X = \lambda^2 X$

Similarly, $A^3X = \lambda^3X$.

Continuing in this way $A^nX = \lambda^nX$

$\lambda^n$is a eigen value of A^n, hence proved

$$A^2 + 2A + 5I = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -6 \\ 0 & 4 & 20 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 2 & -4 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 6 & -10 \\ 0 & 13 & 28 \\ 0 & 0 & 20 \end{bmatrix}$$

The Characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 13-\lambda & 6 & -10 \\ 0 & 13-\lambda & 28 \\ 0 & 0 & 20-\lambda \end{vmatrix} = 0$$

$(13-\lambda)[(13-\lambda)(20-\lambda) - 0] - 6\,[0-0] - 10[0-0] = 0$

$(13-\lambda)(13-\lambda)(20-\lambda) = 0$

λ = 13, 13, 20

Hence the eigen values of $A^2 + 2A + 5I$ are 13, 13 and 20.

(b) The probability density function of a random variable X is $f(x) = kx^2(1 - x^3),\ 0 \leq x \leq 1$. Find k, expectation and variance of x. **(5M)**

Solution:

We have $\int_0^1 kx^2(1 - x^3).dx = 1$

$\int_0^1 k(x^2 - x^5).dx = 1$

$k\left(\left[\frac{x^3}{3} - \frac{x^6}{6}\right]\Big| x = 0\ to\ 1\right) = 0$

$k\left[\frac{1}{3} - \frac{1}{6}\right] = 1$

$k.\frac{1}{6} = 1$

k = 6

Mean $\bar{x} = E(x) = \int_0^1 x\, f(x)dx$

$= \int_0^1 6x^2(1 - x^3).dx$

$= 6\int_0^1 x(x^2 - x^5).dx$

$= 6\int_0^1 (x^3 - x^6).dx$

$= 6\left(\left[\frac{x^4}{4} - \frac{x^7}{7}\right]\Big| x = 0\ to\ 1\right)$

$= 6\left[\frac{1}{4} - \frac{1}{7}\right]$

$= \frac{18}{28} = \frac{9}{14}$

E$(x^2) = \int_0^1 x^2\, f(x)dx = \ = \int_0^1 6x^2[x^2(1 - x^3)].dx$

$= 6\int_0^1 x^2(x^2 - x^5).dx$

$= 6\int_0^1 (x^4 - x^7).dx$

$= 6\left(\left[\frac{x^5}{5} - \frac{x^8}{8}\right]\Big| x = 0\ to\ 1\right)$

$= 6\left[\frac{1}{5} - \frac{1}{8}\right]$

$= \frac{18}{40} = \frac{9}{20}$

Variance = E$(x^2) - [E(x)]^2$

$= \frac{9}{20} - \frac{81}{196} = \frac{441 - 406}{980}$

$= \frac{9}{245}$

(c) A machine is set to produce metal plates of thickness 1.5 cm with standard deviation 0.2 cm. A sample 100 plates produced by the machine gave an thickness of 1.52 cm. Is the machine fulfilling the purpose? (5M)

Solution:

(i)The null hypothesis H_o: $\mu = 1.5$
Alternative hypothesis H_a : $\mu \neq 1.5$

(ii)Calculation of test statistic:

Since sample size is large $Z = Z_{cal} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{1.52-1.5}{0.2/\sqrt{100}} = 1$

(iii) Level of significance: α =0.05

(iv) Critical value: the value of $Z\alpha$ at 5% level of significance = 1.96

(v) Decision: since the calculated value of $|Z|$ =1 is less than the table value $Z\alpha$=1.96. Therefore, the null hypothesis is accepted i.e. The machine fulfilling the purpose.

(d) Write the dual of the given LPP:

Minimise $Z=2x_1 + 3x_2 + 4x_3$

Subjected to: $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 = 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_3 \geq 0$ and x_2 is unrestricted. (5M)

Solution:

Minimise $Z=2x_1 + 3x_2 + 4x_3$

Subjected to: $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \geq 3$

$- 3x_1 - x_2 - 7x_3 \geq -3$

$- x_1 - 4x_2 - 6x_3 \geq -5$

Since x_2 is unrestricted, we put $x_2 = x_2' - x_2''$

Minimise $Z=2x_1 + 3x_2' - 3x_2'' + 4x_3$

Subjected to: $2x_1 + 3x_2' - 3x_2'' + 5x_3 \geq 2$

$3x_1 + x_2' - x_2'' + 7x_3 \geq 3$

$- 3x_1 - x_2' + x_2'' - 7x_3 \geq -3$

$- x_1 - 4x_2' + 4x_2'' - 6x_3 \geq -5$

If y_1, y_2', y_2'', y_3 are the dual variables and w is the function of the dual then dual of the given problem will be

Maximise $w= 2y_1 + 3y_2' - 3y_2'' - 5y_3$

Subjected to: $2y_1 + 3y_2' - 3y_2'' - y_3 \leq 2$

$3y_1 + y_2' - y_2'' - 4y_3 \leq 3$

$-3y_1 - y_2' + y_2'' + 4y_3 \leq -3$

$5y_1 + 7y_2' - 7y_2'' - 6y_3 \leq 4$

Putting $y_2' - y_2'' = y_2$, we get

Maximise $w= 2y_1 + 3y_2 - 5y_3$

Subjected to: $2y_1 + 3y_2 - y_3 \leq 2$

$3y_1 + y_2 - 4y_3 \leq 3$

$-3y_1 - y_2 + 4y_3 \leq 3$

$5y_1 + 7y_2 - 6y_3 \leq 4$

$y_1, y_3 \geq 0$ and y_2 is unrestricted.

Q2) (a) Check whether the given matrix A is diagonalizable, diagonalize if it is, where

A = $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$**.** **(6M)**

Solution:

The characteristic equation of A is

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$(-9-\lambda)[(3-\lambda)(7-\lambda)-32] - 4(-56+8\lambda+64) + 4(-64+48-16\lambda) = 0$

$\lambda^3 + \lambda^2 + 5\lambda + 3 = 0$

$-(\lambda+1)(\lambda^2 - 2\lambda - 3) = 0$

$\lambda= -1, \lambda=-1, \lambda= 3$

for $\lambda= -1$,

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1/(-8)$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - (-8)R_1$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - (-16)R_1$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 0$$

The rank of coefficient matrix is 1. The number of unknowns is 3. Hence, there are 3-1 = 2 linearly independent solution. Putting $x_2 = 2t$ and $x_3 = 2s$ then $x_1 = t + s$.

$$X_1 = \begin{bmatrix} t+s \\ 2t \\ 2s \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Corresponding to the eigenvalue 2, we get the following two linearly independent eigenvectors.

$X_1 = [1 \quad 2 \quad 0]'$ and $X_2 = [1 \quad 0 \quad 2]'$

for λ= 3,

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1/(-12)$

$$\begin{bmatrix} 1 & -1/3 & -1/3 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - (-8)R_1$

$$\begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & -8/3 & 4/3 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - (-16)R_1$

$$\begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & -8/3 & 4/3 \\ 0 & 8/3 & -4/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2/(\frac{-8}{3})$

$$\begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & -1/2 \\ 0 & 8/3 & -4/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - (\frac{-8}{2})R_2$

$$\begin{bmatrix}1 & -1/3 & -1/3\\0 & 1 & -1/2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By $R_1-(\frac{-1}{3})R_2$

$$\begin{bmatrix}1 & 0 & -1/2\\0 & 1 & -1/2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1+0x_2-\frac{1}{2}x_3=0$$

$$0x_1+x_2-\frac{1}{2}x_3=0$$

So, $x_1=(1/2)x_3;\ x_2=(1/2)x_3$ and $x_3=x_3$

$$X_1=\begin{bmatrix}1/2\\1/2\\1\end{bmatrix}$$

Thus, A is diagonalised to $\begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 3\end{bmatrix}$ and the diagonalizing matrix is $\begin{bmatrix}1 & 1 & 1/2\\2 & 0 & 1/2\\0 & 2 & 1\end{bmatrix}$.

(b) Verify Green's theorem for $\overline{F}=(x^2-y)i+(2y^2+x)j$ where C is the boundary of region bounded by $y=x^2, y=4$. **(6M)**

Solution:

By Green's Theorem

$$\int_c P.dx+Q.dy=\iint_R \frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}.dx.dy$$

$\int_c P.dx+Q.dy=\int_c (x^2-\text{y})dx+(2y^2+x)dy$

Here, P = x^2- y ; Q = $2y^2+x$

$\frac{\delta Q}{\delta x}=1$; $\frac{\delta P}{\delta y}=-1$

Along C1, y = x^2 and dy = 2x. dx

$\int_c P.dx+Q.dy=\int_0^2[(x^2-x^2)+(2x^4+x).2\text{x}].dx$

$=\int_0^2(4x^5+2x^2).dx=\left(\frac{4x^6}{6}+\frac{2x^3}{3}\middle|x=0\ to\ 2\right)$

$=\frac{256}{6}+\frac{16}{3}=\frac{144}{3}$

Along C2, y = 4 and dy = 0

$\int_c P.dx+Q.dy=\int_2^0(x^2-4).dx$

$=\left(\frac{x^3}{3}-4x\middle|x=2\ to\ 0\right)$

$$= 0 - (\frac{8}{3} - 8) = \frac{16}{3}$$

Along C3, x = 0 and dx = 0

$$\int_c P.dx + Q.dy = \int_4^0 (2y^2)dy$$
$$= \left(0 - \frac{2y^3}{3}\middle| x = 4 \; to \; 0\right)$$
$$= 0 - (\frac{128}{3}) = \frac{-128}{3}$$

$$\int_c P.dx + Q.dy = \frac{144}{3} + \frac{16}{3} - \frac{128}{3} = \frac{32}{3} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\iint_R \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y} .dx.dy \int_0^4 \int_0^{\sqrt{y}} 2.dx.dy$$
$$= \int_0^1 (2x|x = 0 \; to \; \sqrt{y}).dy$$

$$= 2\int_0^4 \sqrt{y}.dy$$

$$= \left(2 * \frac{y^{3/2}}{3/2}\middle| x = 0 \; to \; 4\right) = \frac{32}{3} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii), the theorem is proved.

(c) The heights of six randomly chosen sailors are in inches :63,65,68,69,71 and 72. The heights of ten randomly soldiers are :61,62,65,66,69,69,70,71,72 and 73. Discuss in the light that these data throw on the suggestion that the soldiers on an average taller than sailors. (8M)

Solution:

We first calculate the mean and standard deviation of the heights of both sailors and soldier

Sailors			Soldiers		
Height X_1	d_1 $(x_1 - \overline{x_1})$	d_1^2 $(x_1 - \overline{x_1})^2$	Height X_2	d_2 $(x_2 - \overline{x_2})$	d_2^2 $(x_2 - \overline{x_2})^2$
63	-5	25	61	-6.8	46.24
65	-3	9	62	-5.8	33.64
68	0	0	65	-2.8	7.84
69	1	1	66	-1.8	3.24
71	3	9	69	1.2	1.44
72	4	16	69	1.2	1.44
			70	2.2	4.84
			71	3.2	10.24
			72	4.2	17.84
			73	5.2	27.04
$\sum x_1 = 408$	0	$\sum(x_1 - \overline{x_1})^2 = 60$	$\sum x_2 = 678$	0	$\sum(x_2 - \overline{x_2})^2 = 153.60$

Now,

$$X_1 = \frac{\sum X_1}{N} = \frac{408}{6} = 68, \quad X_2 = \frac{\sum X_2}{N} = \frac{678}{10} = 67.8$$

The unbiased estimate of the common population

$s_p = \sqrt{\frac{\sum(X_1-\overline{X_1})^2+\sum(X_2-\overline{X_2})^2}{n_1+n_2-2}} = \sqrt{\frac{60+153.6}{6+10-2}} = \sqrt{15.26} = 3.9$

Null Hypothesis Ho: $\mu_1 = \mu_2$

Alternative Hypothesis Ha: $\mu_1 \neq \mu_2$

Calculation of test statistic

$t = \frac{\overline{X_1}-\overline{X_2}}{S.E.}$

Now, $\overline{X_1} = 68$, $\overline{X_2} = 67.8$

S.E. = $s_p * \sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 3.9 * \sqrt{\frac{1}{6}+\frac{1}{10}} = 2.014$

$t = \frac{\overline{X_1}-\overline{X_2}}{S.E.} = \frac{68-67.8}{2.014}$

Level of significance: $\alpha = 0.05$

Critical value: The value of t at $\alpha = 0.05$ for v = 6 + 10 – 2=14 degrees of freedom is $t_\alpha = 2.145$

Decision: Since the computed value | t | = 0.099 is smaller than the table value $t_\alpha = 2.145$, the hypothesis is accepted.

Therefore, the means are equal i.e. the suggestion that the soldiers on the average are taller than sailors cannot be accepted.

Q3) (a) Use Big-M method to solve

Minimise $z = 10x_1 + 3x_2$

Subjected to: $x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1, x_2 \geq 0$ **(6M)**

Solution:

Maximise $z' = -z = -10x_1 - 3x_2$

Subjected to: $x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$z' = -10x_1 - 3x_2 - 0s_1 - 0s_2 - MA_1 - MA_2$

$x_1 + 2x_2 - s_1 - 0s_2 + A_1 + 0A_2 \geq 3$

$x_1 + 4x_2 - 0s_1 - s_2 + 0A_1 + A_2 \geq 4$

We now eliminate - MA_1 and - MA_2 from the object function by adding M times the first and second constraints to the object function.

$z' = -10x_1 - 3x_2 - 0s_1 - 0s_2 - MA_1 - MA_2 + -10x_1 - 3x_2 + Mx_1 + 2Mx_2 - Ms_1 + MA_1 – 3M + Mx_1 + 4Mx_2 - Ms_2 + MA_2 - 4M$

$z' = (-10 + 2M)\, x_1 + (-3 + 6M)\, x_2 - Ms_1 - Ms_2 + 0A_1 + 0A_2 - 7M$

$z' + (10 - 2M)\, x_1 + (3 - 6M)\, x_2 + Ms_1 + Ms_2 + 0A_1 + 0A_2 = 7M$

and constraints as above

Setting $x_1 = 0, x_2 = 0, s_1 = 0, s_2 = 0$, we have $A_1 = 3, A_2 = 4$

Iteration No.	Basic Var.	Coefficient Of						R.H.S. Soln	Ratio
		x_1	x_2	s_1	s_2	A_1	A_2		
0	z’	10-2M	3-6M	M	M	0	0	-7M	
A_2 leaves	A_1	1	2	-1	0	1	0	3	1.5
x_1 enters	A_2	1	4*	0	-1	0	1	4	1
1	z’	37/4-M/2	0	M	3/4- M/2	0		-M-3	
A_1 leaves	A_1	1/2*	0	-1	½	0		1	2
s_2 enters	x_2	1/4	1	0	-1/4	0		1	4
2	z’	17/2	0	3/2	0			-9/2	
	s_2	1	0	-2	1			2	
	x_2	1/2	0	-1/2	0			3/2	

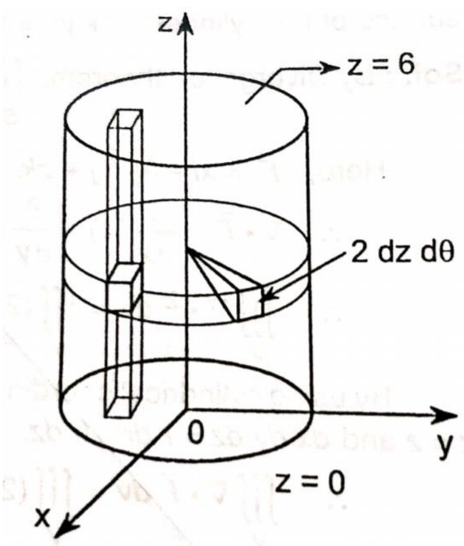
$x_1 = 0$ $\quad x_2 = 3/2$ $\quad z' = -9/2$ $\quad z = 9/2$

(b) Using Gauss Divergence Theorem, evaluate $\iint_S \bar{N}.\bar{F}$ where S is the surface of the region bounded by cylinder $x^2 + y^2 = 4, z = 0, z = 6$ and $\bar{F} = 2xi + xyj + zk$. **(6M)**

Solution:

By divergence formula,

$\iint_S \bar{F}.d\bar{S} = \iiint_V \nabla.\bar{F}.div$



Now, $\bar{F} = 2xi + xyj + zk$

$\nabla.\bar{F} = \frac{\delta(2x)}{\delta x} + \frac{\delta(xy)}{\delta y} - \frac{\delta(z)}{\delta z}$

$= 2 + x + 1$

$= 3 + x$

$$\iiint_V \nabla.\bar{F}.div = \iiint_V (3+x).dv = \iiint_V (3+x).dx.dy.dz$$

Now, to cover the whole volume bounded by the cylinder $x^2 + y^2 = 4, z = 0$ and $z = 6$, z varies from 0 to 6, y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$, and x varies from -2 to 2

$$\iiint_V (3+x)\,.\,dx.\,dy.\,dz = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{6}(3+x).\ dxdydz$$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(3z + xz|z = 0\ to\ 6)\ .dx\,dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 18 + 6x\ .dx\,dy$

$= \int_{-2}^{2}\left(18y + 6xy\middle|x = -\sqrt{4-x^2}\ to\ \sqrt{4-x^2}\right)$

$= \int_{-2}^{2} 18\sqrt{4-x^2} + 6x\sqrt{4-x^2} - (-18\sqrt{4-x^2} - 6x\sqrt{4-x^2})\,.\,dx$

$= \int_{-2}^{2} 18\sqrt{4-x^2} + 12x\sqrt{4-x^2}\,.\,dx$

$= \left(36\left(\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right) - 4(4-x^2)^{3/2}\middle|x = -2\ to\ 2\right)$

$= 72\pi$

(c) Find the rank, index, signature and class of the following Quadratic Form by reducing it to its canonical form using Congruent transformations $4x^2 + 3y^2 + 12z^2 - 8xy + 16yz - 20xz$. **(8M)**

Solution:

The matrix form is

$$A=\begin{bmatrix} 4 & -4 & -10 \\ -4 & 3 & 8 \\ -10 & 8 & 12 \end{bmatrix}$$

We write A=IAI

$$\begin{bmatrix} 4 & -4 & -10 \\ -4 & 3 & 8 \\ 10 & 8 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + R_1,\ R_3 + \frac{10}{4}R_1,\ C_2 + C_1,\ C_3 + \frac{10}{4}C_1$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 10/4 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 10/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + 2R_2, C_3 - 2C_2$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 9/2 & 2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 9/2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9/2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x_1 = y_1 + y_2 + \frac{9}{2}y_3$

$x_2 = -y_2 + 2y_3$

$x_3 = y_3$

The rank = 3, index = 2

Signature = difference between positive squares and negative squares = 2 – 1 = 1

Since some diagonal elements are positive, some are negative, the value class is indefinite.

Q4 (a) The number of accidents in a year attributed to taxi drivers in a city follow Poisson distribution with mean 3. Out of 1000 taxi drivers, with (i) no accidents in a year (ii) more than 3 accidents in a year. **(6M)**

Solution:

P (X = x) = $\frac{e^{-m}m^x}{x!}$, x = 0, 1, 2, ……

We are given m = 3

P (X = x) = $\frac{e^{-3}3^x}{x!}$, x = 0, 1, 2, ……

P (X = 0) = $\frac{e^{-3}3^0}{0!} = 0.0498$

P (X = 1) = $\frac{e^{-3}3^1}{1!} = 0.1494$

P (X = 2) = $\frac{e^{-3}3^2}{2!} = 0.2241$

P (X = 3) = $\frac{e^{-3}3^3}{3!} = 0.2241$

Expected number of drivers with no accidents = N x p (0) = 1000 x 0.0498 = 49.8 = 50 nearly

p (0, 1, 2 accidents) = p(0) + p(1) + p(2) = 0.0498 + 0.1494 + 0.2241 = 0.4233

p(more than 3 accidents) = 1 – 0.4233 = 0.5767

Expected number of drivers with more than 3 accidents = N x p = 1000 x 0.5676

= 576.7 = 577 nearly.

(b) Verify Cayley Hamilton Theorem and hence find A⁻¹, if A = $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$**.** **(6M)**

Solution:

The characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)[(2-\lambda)^2 - 1] + 1\,[-1(2-\lambda)] + 1[1-(2-\lambda)] = 0$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Cayley Hamilton Theorem states this equation is satisfied by A

$A^3 - 6A^2 + 9A - 4I = 0$

Now, $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now multiply the equation by A^{-1},

$4A^{-1} = (A^2 - 6A + 9I)$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

(c) In a test given to two groups of students drawn from two normal populations marks obtained were as follows

Group A:18, 20, 36, 50, 49, 36, 34, 49, 41

Group B:26, 28, 26, 35, 30, 30, 44, 46

Examine the equality of variances (Given: $F_{0.025} = 5.6$ with d.f. 8 & 6 and $F_{0.025} = 4.65$ with d.f. 6 & 8.)

Solution: **(8M)**

We first calculate the mean and standard deviation of the heights of both sailors and soldier

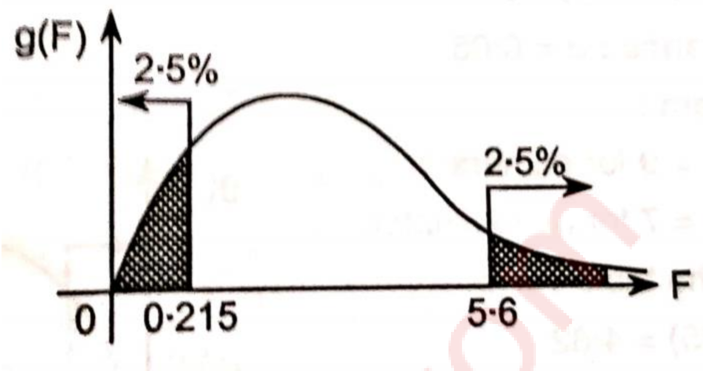
Group A			Soldiers		
x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$
18	-19	396	29	-5	25
20	-17	289	28	-6	36
36	-1	1	26	-8	64
50	13	169	35	1	1
49	12	144	30	-4	16
36	-1	1	44	10	100
34	-3	9	46	12	144
49	12	144			
41	4	16			
$\sum x = 333$	0	$\sum (x-\bar{x})^2 = 1134$	$\sum y = 238$	0	$\sum (y-\bar{y})^2 = 386$

$\bar{x} = \frac{333}{9} = 37,\ \bar{y} = \frac{238}{7} = 34$

$\sum(x_i - \bar{x})^2 = 1134$, $\sum(y_i - \bar{y})^2 = 386$

Null Hypothesis (H_o): ${\sigma_1}^2 = {\sigma_2}^2$

Alternate Hypothesis (H_a): ${\sigma_1}^2 \neq {\sigma_2}^2$



Calculation of test statistic

$F = \frac{n_1 {s_1}^2/(n_1-1)}{n_2 {s_2}^2/(n_2-1)}$

But $n_1 {s_1}^2 = \sum(x_i - \bar{x})^2$ and $n_2 {s_2}^2 = \sum(y_i - \bar{y})^2$

$F = \frac{1134/8}{386/6} = 2.203$

Level of significance: α = 0.05

Degrees of freedom: $v_1 = n_1 - 1 = 9 - 1 = 8$ for the numerator

$v_2 = n_2 - 1 = 7 - 1 = 6$ for the denominator

Critical Value: The table value

$F_{(8,6)}(0.025) = 5.6$
$F_{(6,8)}(0.025) = 4.65$

And $\frac{1}{F_{(6,8)}(0.025)} = \frac{1}{4.65} = 0.215$

Decision: Since the calculation value F = 2.203 lies between 0.215 and 4.65, we accept the null hypothesis

Q5) (a) Show that A = $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is derogatory and hence find minimal polynomial. **(6M)**

Solution:

The characteristic equation of A is

$|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$

(5 – λ)*[(4 – λ)*(–4 – λ) + 12] + 6 [4 + λ - 6] – 6 [6 – 3(4 – λ)] = 0

(5 – λ)*(–4 – λ^2) + 6 [– 2 + λ] – 6 [–6 – 3λ] = 0

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 2)(\lambda - 2)(\lambda - 1) = 0$

λ = 2, 2, 1

Let us now find minimal polynomial of A. We know that each characteristic root of A is a root of the minimal polynomial of A. So if f(x) is the minimal polynomial of A, then (x-2) and (x-1) are the factors of f(x) = $x^2 - 3x + 2$

Let us see whether $(x-2)(x-1) = x^2 - 3x + 2$ annihilates A

$$A^2 - 3A + 2I = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -18 & -18 \\ -2 & 10 & 2 \\ 9 & -18 & -14 \end{bmatrix} - \begin{bmatrix} 15 & -18 & -18 \\ -3 & 12 & 6 \\ 9 & -18 & -12 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, f(x) is monic polynomial of lowest degree that annihilates A. Hence f(x) is minimal polynomial of A. Since its degree is less than the order of A, A is derogatory.

(b) Prove that $\bar{F} = 2xyz^2 i + (x^2z^2 + z\cos yz)j + (2x^2yz + y\cos yz)k$ is a conservative field. Find ϕ such that $\bar{F} = \nabla.\phi$. Hence find the work done in moving an object in this field from (0,0,1) to (1,π/4,2).

Solution: **(6M)**

$$\text{Curl }(\bar{F}) = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$= (2x^2z + \cos yz - yz\sin yz - 2x^2z + yz\sin yz - \cos yz)i + (4xyz - 4xyz)j \quad + (2xz^2 - 2zx^2)k$
= 0

$\bar{F}$ is irrotatonal.

Since $\bar{F}$ is irrotatonal there exists a scalar function ϕ, such that $\bar{F} = \nabla.\phi$

$$2xyz^2 i + (x^2z^2 + z\cos yz)j + (2x^2yz + y\cos yz)k = \frac{\delta\phi}{\delta x} + \frac{\delta\phi}{\delta y} + \frac{\delta\phi}{\delta z}$$

$\frac{\delta\phi}{\delta x} = 2xyz^2$; $\frac{\delta\phi}{\delta y} = (x^2z^2 + z\cos yz)$; $\frac{\delta\phi}{\delta z} = (2x^2yz + y\cos yz)$

dϕ = $\frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz$

$= 2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$
$= (2xyz^2dx + x^2z^2dy + 2x^2yzdz) + (z\cos yz\,dy + y\cos yz\,dz)$
$= d(x^2yz^2 + \sin yz)$

Φ=$x^2yz^2 + \sin yz$

Now, Work done = $\int_c \bar{F}.d\bar{r} = \int_c d(x^2yz^2 + \sin yz$

$= \left(x^2yz^2 + \sin yz \middle| (0, 0, 1) to \left(1, \frac{\pi}{4}, 2\right)\right)$

= π +1

(c) Out of a sample 120 persons in a village, 76 were administered a new drug for preventing influenza and out of them 24 persons were attacked by influenza. Out of these were not administered the new drugs, 12 persons were not affected by influenza. Use chi-square method to find out whether the new drug is effective or not? (8M)

Solution:

The above data can be arranged in the following 2 x 2 contingency table

New drug	Effect of	Influenza	Total
	Attacked	Not attacked	
Administered	24	76 – 24 = 52	76
Not administered	44 – 12 = 32	12	120 – 76 =44
Total	120 – 64 = 56 24 + 32 = 56	52 + 12 = 64	120

Null Hypothesis: 'Attack of influenza' and the administration of the new drug are independent

Computation of statistic:

$$x_o{}^2 = \frac{N(ad-bc)^2}{(a+c)(b+d)(a+b)(c+d)}$$

$$= \frac{120(24*12-52*32)^2}{56*64*76*44}$$

$$= \frac{120*1376^2}{54*64*76*44} = 18.95$$

Expected value:

$x_e{}^2 = \sum\left(\frac{(O-E)^2}{E}\right)$ follows x^2 distribution with (2 - 1) x (2 -1) d.f. = 3.84

Inference: Since $x_o{}^2 = x_e{}^2$, Ho is rejected at 5% level of significance. Hence we conclude that the new drug is definitely effective in controlling (preventing) the disease (influenza).

Q6) (a) Evaluate $\int_c (\boldsymbol{x+2y})\boldsymbol{dx} + (\boldsymbol{x-z})\boldsymbol{dy} + (\boldsymbol{y-z})\boldsymbol{dz}$ where C is the boundary of the triangle with vertices (2,0,0),(0,3,0),(0,0,6) oriented in the anticlockwise direction. (6M)

Solution:

By Stokes theorem $\int_c \bar{F}d\bar{r} = \iint_s \bar{N}.\nabla.\bar{F}ds$

Now, $\nabla X \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x+2y & x-z & y-z \end{vmatrix} = (1+1)i - (0-1)j + (1-2)k$

= 2i + j - 1k

Further ϕ = 3x + 2y + z – 6

Normal to the plane ABC,

$\nabla\phi = \frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k = 3i + 2j + 1k$

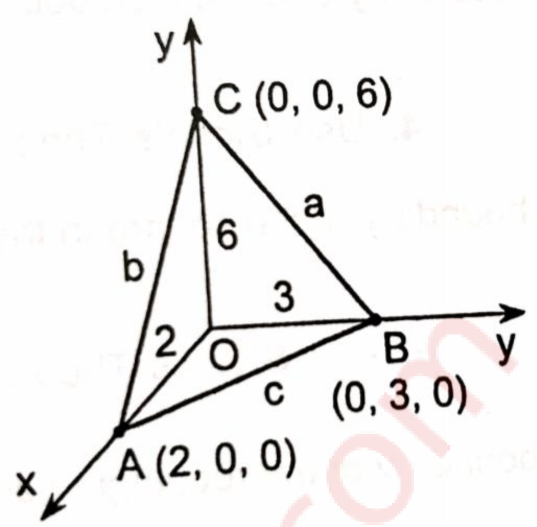
Unit normal to the plane ΔABC

$\overline{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3i+2j+1k}{\sqrt{14}}$

$\iint_c \ \overline{N}.\nabla \ X \ \overline{F}. \ ds = \iint_c \left(\frac{3i+2j+1k}{\sqrt{14}}\right)(2i + j - k).\text{ds}$

$= \iint_c \ \frac{3*2+2*1+1*(-1)}{\sqrt{14}}.ds$

$= \iint_c \ \frac{7}{\sqrt{14}}.ds$



The equation of the plane is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$,

OA = 2, OB = 3, OC = 6

But $\iint \ ds$ over the triangle ABC is the area of the triangle ABC. If AB = c, BC = a, CA = b and θ is the angle between AB and BC, then the area of ΔABC = $\frac{1}{2}ac \ \sin\theta$.

By cosine rule, $b^2 = a^2 + c^2 - 2\text{ac} \ \cos\theta$

Now, $a^2 = 36 + 9 = 45$, $b^2 = 36 + 4 = 40$, $c^2 = 9 + 4 = 13$

$40 = 45 + 13 - 2 * \sqrt{45} * \sqrt{13} \ \cos\theta$

$$\cos\theta = \frac{18}{2 * \sqrt{45} * \sqrt{13}} = \frac{9}{\sqrt{45} * \sqrt{13}}$$

$$\sin\theta = \sqrt{1 - \cos\theta^2} = \sqrt{1 - \left(\frac{9}{\sqrt{45} * \sqrt{13}}\right)^2} = \sqrt{\frac{504}{45 * 13}}$$

area of ΔABC = $\frac{1}{2}ac \ \sin\theta = 0.5 * \sqrt{45} * \sqrt{13} * \sqrt{\frac{504}{45*13}} = \sqrt{126}$

$\iint \ ds = \sqrt{126}$

$\iint_c \ \overline{N}.\nabla \ X \ \overline{F}. \ ds = \frac{7\sqrt{126}}{\sqrt{14}} = 7\sqrt{9} = 21$

(b) Ten individual are chosen at random from a population and their heights are found to be (inches): 63, 63, 66, 67, 68, 69, 70, 71 and 71. In the light of the data, discuss the suggestion that the mean height in the population is 66 inches. (6M)

Solution:

N = 10 (<30, so it is small sample)

Null Hypothesis (H_o) : μ = 65

Alternate Hypothesis (H_a) : μ! = 65 [two tailed test]

LOS = 5 % (two tailed test)

Degree of freedom = n – 1 = 10 – 1 = 9

Critical value (t_α) = 2.2622

Values (x_i)	$D_i = x_i - 67$	D_i^2
63	-4	16
63	-4	16
64	-3	9
65	-2	4
66	-1	1
69	2	4
69	2	4
70	3	9
70	3	9
71	4	16
Total	0	88

$\bar{d} = \frac{\sum d_i}{n} = \frac{0}{10} = 0$

$\bar{x} = a + \bar{d} = 67 + 0 = 67$

Since sample is small, $s = \sqrt{\frac{\sum d_i^2}{n} - (\sqrt{\frac{\sum d_i}{n}})^2}$

$$= \sqrt{\frac{88}{10} - (\sqrt{\frac{0}{10}})^2} = 2.9965$$

$$S.E. = \frac{s}{\sqrt{n-1}} = \frac{2.9965}{\sqrt{10-1}} = 0.9888$$

Step 4: Test statistic

$$t_{cal} = \frac{\bar{x} - \mu}{S.E.} = \frac{67 - 65}{0.9888} = 2.0227$$

Step 5: Decision

Since $|t_{cal}| < t_x$, H_0 is accepted.
The mean height of the universe is 65 inches.

(c) Using dual simplex method solve the given LPP

Minimise $z=2x_1+x_2$

Subjected to: $3x_1+x_2 \leq 3$,

$4x_1+3x_2 \geq 6$,

$x_1+2x_2 \leq 3$,

$x_1, x_2 \geq 0$ **(8M)**

Solution:

Minimise $z = 2x_1 + x_2$

Subjected to: $3x_1 + x_2 \leq 3$

$-4x_1 - 3x_2 \leq -6$,

$x_1 + 2x_2 \leq 3$.

Introducing the slack variables s_1, s_2, s_3.

Maximise $z = 2x_1 + x_2 - 0s_1 - 0s_2 - 0s_3$

$z - 2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3$

Subjected to: $3x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 3$

$-4x_1 - 3x_2 + 0s_1 + s_2 + 0s_3 = -6$,

$x_1 + 2x_2 + 0s_1 + 0s_2 + s_3 = 3$.

Iteration Number	Basic Variables	Coefficient of					R.H.S Solution
		x_1	x_2	s_1	s_2	s_3	
0	Z	-2	-1	0	0	0	
	s_1	3	1	1	0	0	3
	s_2	-4	-3*	0	1	0	-6
	s_3	1	2	0	0	1	3
Ratio		-2	1/3	0	0	0	
1	Z	-2	2	0	-1	0	
	s_1	5/3	0	1	1/3	0	1
	x_2	4/3	1	0	-1/3	0	2
	s_3	-5/3*	0	0	2/3	1	-1
Ratio		2/3	0	0	1/3	0	
2	Z	-2	-1	0	0	0	
	s_1	0	0	1	1	1	0
	x_2	0	1	0	0.2	0.8	1.2
	x_1	1	0	0	-0.4	-0.6	0.6