M3 SOLUTION OF QUESTION PAPER

(CBCGS MAY 19)

Q.P. CODE: 58543

$\boldsymbol{Q.1.a)}$ ***Find Laplace transform of*** $f(t) = t\int_0^t e^{-2u}\sin 4u\, du.$ (5)

Solution : $L[\sin 4u] = \frac{4}{s^2+4^2}$

$$\therefore L[e^{-2u}\sin 4u] = \frac{4}{(s+2)^2+16}$$

$$\therefore L[e^{-2u}\sin 4u] = \frac{4}{s^2+4s+20}$$

$$\therefore L\left[\int_0^t e^{-2u}\sin 4u\, du\right] = \frac{1}{s}.\frac{4}{s^2+4s+20}$$

$$\therefore L\left[\int_0^t e^{-2u}\sin 4u\, du\right] = \frac{4}{s^3+4s^2+20}$$

$$\therefore L\left[t\int_0^t e^{-2u}\sin 4u\, du\right] = (-1)^1\frac{d}{ds}\left(\frac{4}{s^3+4s^2+20s}\right)$$

$$= -1\times 4\times\frac{-1}{(s^3+4s^2+20s)^2}.\frac{d}{ds}(s^3+4s^2+20s)$$

$$= \frac{4}{[s(s^2+4s+20)]^2}.(3s^2+8s+20)$$

$$\therefore L\left[t\int_0^t e^{-2u}\sin 4u\, du\right] = \frac{4(3s^2+8s+20)}{s^2(s^2+4s+20)^2}$$

$\boldsymbol{b)}$ ***Show that the set of functions*** $\sin nx, n = 1,2,3\ldots.$ ***is orthogonal on*** $(0,2\pi)$. (5)

$Solution : Let\ f_n(x) = \sin nx$

$\therefore\ f_m(x) = \sin mx$

$Consider,$

$$\int_a^b f_n(x)\, f_m(x) dx = \int_0^{2\pi} \sin nx \,.\, \sin mx\, dx$$

$$= \frac{1}{2}\int_0^{2\pi} 2 \sin nx \,.\, \sin mx\, dx$$

$$\therefore\ \int_a^b f_n(x)\, f_m(x) dx = \frac{1}{2}\int_0^{2\pi} [\cos(nx - mx) - \cos(nx = mx)] dx\ \rightarrow (1)$$

$Case\ 1 : m \neq n$

$$\therefore\ \int_a^b f_n(x)\, f_m(x) dx = \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left\{\left[\frac{\sin(n-m)2\pi}{n-m} - \frac{\sin(n+m)2\pi}{n+m}\right] - \left[\frac{\sin 0}{n-m} - \frac{\sin 0}{n+m}\right]\right\}$$

$$= \frac{1}{2}\{[0-0] - [0-0]\} = 0$$

$Case\ 2 : Put\ m = n\ in\ (1)$

$$\int_a^b [f_n(x)]^2 dx = \frac{1}{2}\int_0^{2\pi} [\cos(nx - nx) - \cos(nx + nx)]\, dx$$

$$= \frac{1}{2}\int_0^{2\pi} [1 - \cos 2nx]\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left\{\left[2\pi - \frac{\sin 4n\pi}{2n}\right] - \left[0 - \frac{\sin 0}{2n}\right]\right\}$$

$$= \frac{1}{2}\{[2\pi - 0] - [0 - 0]\} = \pi \neq 0$$

$From\ Case\ 1\ and\ 2,$

$The\ set\ of\ functions\ \{\sin nx\}\ n = 1,2,3,\dots\ is\ orthogonal\ w.r.t.(0,2\pi).$

$\because \int_a^b [f_n(x)]^2 dx \neq 1, the\ given\ set\ of\ functions\ are\ Not\ Orthogonal.$

$\boldsymbol{c)\ Calculate\ Spearman's\ rank\ correlation\ coefficient\ R, from\ the\ given\ data,}$

X: 12, 17, 22, 27, 32.

Y: 113, 119, 117, 115, 121.

$Solution$:

X	Y	R1	R2	d1=R1-R2	${d_I}^2$
12	113	5	5	0	0
17	119	4	2	2	4
22	117	3	3	0	0
27	115	2	4	-2	4
32	121	1	1	0	0
				Total	8

$Here, n = 5$

$$\therefore Spearman's\ rank\ correlation\ Co-officient\ R = 1 - \frac{6\sum {d_I}^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 8}{5 \times (5^2 - 1)}$$

$$= 1 - 0.4$$

$$= 0.6$$

$\therefore Spearman's\ rank\ correlation\ Co-officient = 0.6$

$\boldsymbol{d)}$ ***Find the constants a, b, c, d, e if***

$$\boldsymbol{f(t) = ax^3 + bxy^2 + 3x^2 + cy^2 + x + i(dx^2y - 2y^3 + exy + y)}$$

is analytic. **(5)**

$Solution: Let\ f(z) = u + iv$

$= (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$

Comparing Real and Imaginary parts on both sides, we get

$u = ax^3 + bxy^2 + 3x^2 + cy^2 + x$ *and* $v = dx^2y - 2y^3 + exy + y$

Partially differentiating w.r.t.' x' we get,

$u_x = 3ax^2 + by^2 + 6x + 0 + 1$ *and* $v_x = 2dxy - 0 + ey + 0$

Partially differentiating w.r.t.' y' we get,

$u_y = 0 + 2bxy + 0 + 2cy + 0$ *and* $v_y = dx^2 + 6y^2 + ex + 1$

Since $f(z)$ *is analytic, by Cauchy Reiman's equations,*

$u_x = v_y$ *and* $u_y = -v_x$

$\therefore\ 3ax^2 + by^2 + 6x + 1 = dx^2 - 6y^2 + ex + 1$ *and*

$2bxy + 2cy = -(2dxy + ey)$

Comparing coefficients of x and y in like terms, we get,

$3a = d \rightarrow (1)$

$b = -6 \rightarrow (2)$

$e = 6 \rightarrow (3)$

$2b = -2d$

$\therefore d = -b$

$\therefore d = 6 \rightarrow (4)(From\ 2)$

$\therefore From\ (1)\ \&\ (4),\qquad 3a = 6$

$\therefore a = 2$

$2c = -e$

$\therefore 2c = -6 \ \ (From\ 3)$

$\therefore c = -3$

$\therefore a = 2, b = -6, c = -3, d = 6, c = 9$

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2. a) Find Laplace transform of the periodic function, defined as

$$\boldsymbol{f(t)} = \begin{cases} \boldsymbol{t}, & \boldsymbol{0} < t < 1 \\ \boldsymbol{0}, & \boldsymbol{1} < t < 2 \end{cases} \quad \textbf{and } \boldsymbol{f(t+2) = f(t)} \textbf{ for } \boldsymbol{t} > 0 \qquad \textbf{(6)}$$

$Solution: f(t+2) = f(t)$

$\therefore Fundamental\ period\ (a) = 2$

$\therefore By\ definition\ of\ Laplace\ transform\ for\ Periodic\ function,$

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} t dt + \int_1^2 0 dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[t.\frac{e^{-st}}{-s} - 1.\frac{e^{-st}}{(-s)^2} \right]_0^1$$

$$= \frac{1}{1-e^{-2s}} \left[e^{-st}\left(\frac{t}{s} + \frac{1}{s^2}\right) \right]_0^1$$

$$= \frac{1}{1-e^{-2s}} \left[e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - e^0\left(0 + \frac{1}{s^2}\right) \right]$$

$$= \frac{1}{1-e^{-2s}} \left[e^{-s}\left(\frac{s+1}{s^2}\right) - \frac{1}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \times \frac{1}{s^2} [-e^{-s}(s+1) + 1]$$

$$= \frac{1 - e^{-s}(s+1)}{s^2(1-e^{-2s})}$$

$b)$ ***If*** $\boldsymbol{v = 3x^2y + 6xy - y^3}$***, show that*** $\boldsymbol{v}$ ***is harmonic and find the corresponding analytic function*** $\boldsymbol{f(z)u + iv}$. **(6)**

Solution :

$$\frac{\partial v}{\partial x} = 6xy + 6y, \qquad \frac{\partial^2 v}{\partial x^2} = 6y$$

$$\frac{\partial v}{\partial y} = 3x^2 + 6x - 3y^2, \quad \frac{\partial^2 v}{\partial y^2} = -6y$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0.$$

$\therefore$ *v is satisfies Laplace's equation.*

Now we use Milne – Thompson Method.

$$\therefore v_x = 6xy + 6y, \qquad \Psi_2(\mathrm{z}, 0) = 0$$

$$v_y = 3x^2 + 6x - 3y^2, \qquad \Psi_1(\mathrm{z}, 0) = 3\mathrm{z}^2 + 6\mathrm{z}$$

$$\therefore f'(z) = \Psi_1(\mathrm{z}, 0) + \mathrm{i}\Psi_2(\mathrm{z}, 0) = (3\mathrm{z}^2 + 6\mathrm{z}) + 0$$

$$\therefore f(z) = \int (3\mathrm{z}^2 + 6\mathrm{z})\mathrm{dz} = (z^3 + 3z^2) + c$$

$c)$ ***Obtain Fourier series of*** $\boldsymbol{f(x) = x^2}$ ***in*** $\boldsymbol{(0, 2\pi)}$***. Hence, deduce that***

$$\boldsymbol{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + + \cdots} \qquad \textbf{(8)}$$

Solution : Let $c = 0$ *and* $c + 2l = 2\pi$

$$\therefore 0 + 2l = 2\pi$$

$$\therefore l = \pi$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{2^3\pi^3}{3} - 0\right]$$

$$a_0 = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{l}\int_c^{c+2} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos\frac{n\pi x}{\pi}dx$$

$$= \frac{1}{\pi}\left[x^2.\frac{\sin nx}{n} - 2x.\frac{-\cos nx}{n^2} + 2.\frac{-\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(4\pi^2\frac{\sin 2nx}{n} + \frac{4\pi\cos 2nx}{n^2} - \frac{2\sin 2nx}{n^3}\right) - \left(0 - 0 + \frac{2\sin 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{4\pi.1}{n^2} + 0 - 0\right]$$

$$\therefore\ a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin\frac{n\pi x}{\pi}dx$$

$$= \frac{1}{\pi}\left[x^2.\frac{-\cos nx}{n} - 2x.\frac{-\sin nx}{n^2} + 2.\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-4\pi^2\frac{\cos 2nx}{n} + 4\pi\frac{\sin 2nx}{n^2} + \frac{2\cos 2n\pi}{n^3}\right) - \left(0 - 0 + \frac{2\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-4\pi^2 . \frac{1}{n} + 0 + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$\therefore\ b_n = \frac{-4\pi}{n}$$

In Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\therefore\ x^2 = \frac{8\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos\frac{n\pi x}{\pi} + \sum_{n=1}^{\infty} \frac{-4\pi}{n}\sin\frac{n\pi x}{l}$$

$$\therefore\ x^2 = \frac{4\pi^2}{3} + 4\left[\frac{\cos 1x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right] - 4\pi\left[\frac{\sin 1x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

Deduction: *Put* $x = \pi$

$$\therefore\ \pi^2 = \frac{4\pi^2}{3} + 4\left[\frac{\cos 1\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \cdots\right] - 4\pi\left[\frac{\sin 1\pi}{1} + \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} + \cdots\right]$$

$$\therefore\ \pi^2 - \frac{4\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right] - 0$$

$$\therefore\ \frac{-\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\therefore\ \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

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3. a) *Using convolution theorem, find the inverse Laplace transform of* $\boldsymbol{F(s) = \frac{1}{s^2(s+5)^2}}$. **(6)**

$$Solution: F(s) = \frac{1}{s^2(s+5)^2} = \frac{1}{(s+5)^2} \times \frac{1}{s^2}$$

$$Let\ \varphi_1(s) = \frac{1}{(s+5)^2}\ ; \quad \varphi_2(s) = \frac{1}{s^2}$$

$$\therefore f_1(t) = L^{-1}\left[\frac{1}{(s+5)^2}\right] = e^{-5t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-5t}t\ (First\ Shifting\ Property)$$

$$f_2(t) = L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$By\ Convolution\ Theorem,$$

$$L^{-1}[\ \varphi_1(s)\varphi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)du$$

$$\therefore\ L^{-1}\left[\frac{1}{(s+5)^2} \times \frac{1}{s^2}\right] = \int_0^t e^{-5t}u \times (t-u)du$$

$$= \int_0^t (ut-u^2)e^{-5t}du$$

$$= \left[(ut-u^2)\frac{e^{-5t}}{-5} - (t-2u) \times \frac{e^{-5u}}{(-5)^2} + (-2) \times \frac{e^{-5u}}{(-5)^3}\right]_0^t$$

$$= \left[-(ut-u^2)\frac{e^{-5u}}{5} - (t-2u)\frac{e^{-5u}}{5^2} + 2 \times \frac{e^{-5u}}{5^3}\right]_0^t$$

$$= \left\{\frac{e^{-5u}}{5^3}[-25(ut-u^2) - 5(t-2u) + 2]\right\}_0^t$$

$$= \frac{e^{-5u}}{125}[-25(t^2-t^2) - 5(t-2t) + 2] -$$

$$\frac{e^{-5u}}{125}[-25(0-0) - 5(t-0) + 2]$$

$$= \frac{e^{-5u}}{125}[-0 + 5t + 2] - \frac{1}{125}\ [-0 + 5t + 2]$$

$$= \frac{1}{125}[e^{-5t}(5t + 2) + 5t - 2]$$

$$\therefore\ L^{-1}\left[\frac{1}{s^2(s+5)^2}\right] = \frac{1}{125}[e^{-5t}(5t + 2) + 5t - 2]$$

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b) *Solve* $\boldsymbol{\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial u}{\partial t} = 0}$, ***subject to the conditions,*** $\boldsymbol{u(0,t) = 0}$, $\boldsymbol{u(1,t) = 3t, u(x,0) = 0, 0 \le x \le 1}$, ***taking*** $\boldsymbol{h = 0.25}$ ***up to 3 seconds only by using Bender − Schmidt method.*** **(6)**

Solution : We are given $h = 0.25$ *and* $a = 16$

$$k = \frac{a}{2}h^2, \qquad k = \frac{16}{2}(0.25)^2 = 0.5$$

Since $h = 0.25$ *and the* x *is* 0 *to* 1.

We divide x *interval into 5 parts.*

$x = 0, 0.25, 0.50, 0.75, 1.$

We also divide the time interval by taking $k = 0.5$ *upto* 3.

$t_0 = 0, t_1 = 0.5, t_2 = 1, t_3 = 1.5, t_4 = 2, t_5 = 2.5, t_6 = 3.$

By data, $u(0,t) = 0$

Hence, $x = 0$ *and for all values of* $t = 0, 0.5, 1, 1.5, 2, 2.5, 3.$

$\therefore\ u(0,t) = 0$

By data, $u(1,t) = 3t$

Hence, we calculate $x = 1$ *and* $t = 0, 0.5, 1, 1.5, 2, 2.5, 3.$

when $x = 1$ *and* $t = 0 \quad u = 0 \qquad \{u(1,0) = 3 \times 0 = 0\}$

when $x = 1$ *and* $t = 0.5 \quad u = 1.5$

when $x = 1$ *and* $t = 1 \quad u = 3$

when $x = 1$ *and* $t = 1.5 \quad u = 4.5$

$when\ x = 1\ and\ t = 2 \quad u = 6$

$when\ x = 1\ and\ t = 2.5 \quad u = 7.5$

$when\ x = 1\ and\ t = 3 \quad u = 9$

$By\ data, u(x,0) = 0$

$Hence, for\ all\ x = 0, 0.25, 0.50, 0.75, 1\ and\ t = 0$

$\therefore\ u(x,0) = 0$

$Thus\ we\ get\ the\ following\ part\ of\ the\ table.$

t \ x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.5	0				1.5
1	0				3
1.5	0				4.5
2	0				6
2.5	0				7.5
3	0				9

$\boldsymbol{Now, we\ use\ Bender\ Schmidt\ formula\ c = \frac{a+b}{2}\ and\ complete}$

$\boldsymbol{the\ table\ as\ given\ below.}$

t \ x	0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.5	0	0	0	0	1.5
1	0	0	0	0.75	3
1.5	0	0	0.375	1.5	4.5
2	0	0.1875	0.75	2.4375	6
2.5	0	0.375	1.3125	3.375	7.5
3	0	0.65265	1.875	4.40625	9

c) ***Using Residue theorem, evaluate,*** **(8)**

1) $\displaystyle\int_0^{2\pi} \frac{d\theta}{17 - 8\cos\theta}$

Solution : *Let* $z = e^{i\theta}$, $d\theta = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{z^2+1}{2z}$

$$\therefore I = \int_c \frac{\frac{dz}{iz}}{17 - 8\left(\frac{z^2+1}{2z}\right)}$$

$$= \int_c \frac{1}{17 - 8\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$= \int_c \frac{1}{\frac{17z - 4z^2 - 4}{z}} \cdot \frac{dz}{iz}$$

$$= \int_c \frac{1}{(-4z^2 + 17z - 4)i} \cdot dz$$

$$= \int_c \frac{1}{-i(4z^2 - 17z + 4)} \cdot dz$$

To find poles such that,

$4z^2 - 17z + 4 = 0$

$4z^2 - 16z - z + 4 = 0$

$4z(z-4) - 1(z-4) = 0$

$(z-4)(4z-1) = 0$

$z - 4 = 0, \qquad 4z - 1 = 0$

$z = 4, \qquad 4z = 1$

$z = 4, \qquad z = \dfrac{1}{4}.$

$Since\ c\ is\ unit\ circle\ |z| = 1\ and\ pole\ z = 4\ is\ outside\ and\ z = \frac{1}{4}\ is\ inside$

$we\ have,$

$Residue\ at\ simple\ pole\ z = z_0 = \frac{1}{4}.$

$Residue\ of\ f(z) at\ z = z_0 = \lim_{z \to z_0} (z - z_0) f(z)$

$$= \lim_{z \to \frac{1}{4}} \left(z - \frac{1}{4}\right) \frac{1}{-1(4z^2 - 17z + 4)}$$

$$= \frac{1}{-i} \lim_{z \to \frac{1}{4}} \left(\frac{4z - 1}{4}\right) \frac{1}{(z - 4)(4z - 1)}$$

$$= \frac{1}{-4i} \lim_{z \to \frac{1}{4}} \frac{1}{(z - 4)}$$

$$= \frac{1}{-4i} \cdot \frac{1}{\left(\frac{1}{4} - 4\right)}$$

$$= \frac{1}{-4i} \cdot \frac{1}{\frac{-15}{4}}$$

$$= \frac{1}{15i}.$$

$By\ Residue\ theorem$

$$\int_c f(z) dz = 2\pi i (sum\ of\ Residue)$$

$$= 2\pi i \left(\frac{1}{15i}\right) = \int_c f(z) dz = \frac{2\pi}{15}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{17 - 8\cos\theta} = \frac{2\pi}{15}$$

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$$2)\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$Solution:\ Let\ I=\int_0^\infty \frac{dx}{(x^2+1)^2}$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^2}$$

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{(z^2+1)^2}$$

$To\ find\ poles\ such\ that$

$$(z^2+1)^2=0$$

$$z^2+1=0,\qquad z^2+1=0$$

$$z^2-i^2=0,\qquad z^2-i^2=0$$

$$(z-i)(z+i)=0,\qquad (z-i)(z+i)=0$$

$$z=i,z=-i\,,z=i,z=-i$$

$Residue\ at\ pole\ \ z_0=i\ of\ order\ two\ (n=2).$

$$Residue\ f(z)\ of\ z=z_0=\frac{1}{(n-1)i}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\ (z-z_0)^n.f(z)$$

$$=\frac{1}{(2-1)i}\lim_{z\to i}\frac{d^{2-1}}{dz^{2-1}}\ (z-i)^2.\frac{1}{2}.\frac{1}{(z^2+1)^2}$$

$$=\frac{1}{1i}\lim_{z\to i}\frac{d}{dz}\ (z-i)^2.\frac{1}{2}.\frac{1}{(z^2-i)^2}$$

$$=\frac{1}{2}\lim_{z\to i}\frac{d}{dz}\ (z-i)^2.\frac{1}{[(z-i)(z+i)]^2}$$

$$=\frac{1}{2}\lim_{z\to i}\frac{d}{dz}\ (z-i)^2.\frac{1}{(z-i)^2(z+i)^2}$$

$$=\frac{1}{2}\lim_{z\to i}\frac{d}{dz}\frac{1}{(z+i)^2}$$

$$= \frac{1}{2} \lim_{z \to i} \frac{-2}{(z+i)^3} \quad \to \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{n}{x^{n+1}}$$

$$= \frac{1}{2} \cdot \frac{-2}{(i+i)^3}$$

$$= \frac{-1}{(2i)^3} = \frac{-1}{2^3 i^3}$$

$$= \frac{-1}{8(-i)} \quad \to i^3 = -i$$

$$= \frac{+1}{8i}$$

$$By\ Residue\ theorem \int_c f(z)dz = 2\pi i\ (sum\ of\ the\ residue)$$

$$I = 2\pi i \left(\frac{1}{8i}\right)$$

$$I = \frac{\pi}{4}$$

$\mathbf{4.\ a)\ Solve\ by\ Crank - Nicholson\ simlplified\ formula}$

$$\frac{\partial^2 y}{\partial x^2} - \frac{\partial u}{\partial t} = 0.\ u(0,t) = 0, u(1,t) = 0, u(x,0) = 100(x - x^2),$$

$\mathbf{with\ h = 0.25\ for\ one - time\ step.}$ **(6)**

$Solution: Here\ we\ have\ a = 1\ and\ h = 0.25$

$To\ use\ Crank - Nicholson\ formula, we\ must\ have$

$k = ah^2 = 1 \times 0.25^2 = 0.0625$

$The\ interval\ of\ x\ is\ 0\ to\ 1.$

$The\ subinterval\ is\ of\ size\ h = 0.25$

$x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$

$By\ data\ u(0,t) = 0$

$When\ x = 0\ , for\ all\ values\ of\ t\ , u = 0, when\ t = 0,1.$

$By\ data\ u(1,t) = 0$

$When\ x = 1,$

$for\ all\ values\ of\ t\ , u = 0, when\ t = 0,1.$

$By\ data\ , u(x,0) = 100\ (x - x^2)$

$When\ x = 0, 0.25, 0.5, 0.75\ , 1.$

$when\ x = 0, u = 100(0 - 0^2) = 0$

$when\ x = 0.25\ , u = 100(0.25 - 0.25^2) = 18.75$

$when\ x = 0.5\ , u = 100(0.5 - 0.5^2) = 25$

$when\ x = 0.75\ , u = 100(0.75 - 0.75^2) = 18.75$

$when\ x = 1\ , u = 100(1 - 1^2) = 0$

t \ x	0	0.25	0.5	0.75	1
0	0	18.75	25	18.75	0
1	0	u_1	u_2	u_3	0

$Now\ by\ Crank - Nicholson\ formula\ we\ calculate\ the\ remaing\ values.$

$$e = \frac{1}{4}(a + b + c + d)$$

$$u_1 = \frac{1}{4}(0 + 0 + 25 + u_2) \quad \rightarrow (1)$$

$$u_2 = \frac{1}{4}(u_1 + 18.75 + 18.75 + u_3\ ) \quad \rightarrow (2)$$

$$u_3 = \frac{1}{4}(u_2 + 25 + 0 + 0) \quad \rightarrow (3)$$

$By\ solving\ these\ equations\ we\ get,$

$u_1 = 9.82\ , u_2 = 14.28, u_3 = 9.82$

t \ x	0	0.25	0.5	0.75	1
0	0	18.75	25	18.75	0
1	0	9.82	14.28	9.82	0

---

b) Evaluate $\displaystyle\int_c \frac{z}{(z-2)(z+1)^2}\,dz,\qquad C: |z| = 3.$ **(6)**

$Solution: Let\ I = \displaystyle\int_c \frac{z}{(z-2)(z+1)^2}\,dz$

$To\ find\ poles\ such\ that,$

$(z-2)(z+1)^2 = 0$

$z = 2\ is\ a\ simple\ pole\ and\ inside\ |z| = 3.$

$z = -1\ is\ a\ pole\ of\ order\ 2\ and\ inside\ |z| = 3.$

$1)\ The\ Residue\ at\ simple\ pole\ z_0 = 2\ is$

$$Residue\ f(z)\ of\ z = z_0 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$Residue\ f(z)\ of\ z = 2 = \lim_{z \to 2} (z-2) \frac{z}{(z-2)(z+1)^2}$$

$$= \lim_{z \to 2} \frac{z}{(z+1)^2}$$

$$= \frac{2}{(2+1)^2}$$

$$Residue\ f(z)\ of\ z = 2 = \frac{2}{9}$$

$2)\ The\ Residue\ of\ pole\ z_0 = -1\ of\ order\ 2.$

$$Residue\ f(z) of\ z = z_0 = \frac{1}{(n-1)i} . \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} . (z - z_0)^n f(z)$$

$$= \frac{1}{1i} \cdot \lim_{z \to -1} \frac{d}{dz} \cdot (z+1)^2 \cdot \frac{z}{(z-2)(z+1)^2}$$

$$= \lim_{z \to -1} \frac{d}{dz} \cdot \frac{z}{z-2}$$

$$= \lim_{z \to -1} \left[\frac{(z-2)1 - z(1-0)}{(z-2)^2} \right]$$

$$= \lim_{z \to -1} \left[\frac{z-2-z}{(z-2)^2} \right]$$

$$= \frac{-2}{(-1-2)^2} = \frac{-2}{9}.$$

By Residue theorem

$I = 2\pi i\ (sum\ of\ Residue)$

$$= 2\pi i \left(\frac{2}{9} - \frac{2}{9} \right)$$

$I = 0$

$$\therefore \int_c \frac{z}{(z-2)(z+1)^2}\, dz = 0$$

---

c) Solve $(D^2 - 2D + 1)y = e^{-t}$ with $y(0) = 2, y'(0) = -1$

where $D = \frac{d}{dt}$. **(8)**

Solution: $(D^2 - 2D + 1)y = e^{-t}$

$\therefore\ D^2y - 2Dy + y = e^{-t}$

Taking Laplace Transform,

$L[D^2y] - 2L[Dy] + L[y] = L[e^{-t}]$

$$\therefore\ [s^2\bar{y} - sy(0) - y'(0)] - 2[s\bar{y} - y(0)] + \bar{y} = \frac{1}{s+1}$$

$$\therefore\ s^2\bar{y} - s(2) - \ (-1) - 2s\bar{y} + 2(2) + \bar{y} = \frac{1}{s+1}$$

$$\therefore\ (s^2 - 2s + 1)\bar{y} = \frac{1}{s+1} + 2s - 1 - 4$$

$$\therefore\ (s-1)^2\bar{y} = \frac{1 + 2s(s+1) - 5(s+1)}{s+1}$$

$$\therefore\ \bar{y} = \frac{1 + 2s^2 + 2s - 5s - 5}{(s-1)^2(s+1)}$$

$$\therefore y = L^{-1}\left[\frac{2s^2 - 3s - 4}{(s-1)^2(s+1)}\right] \ \rightarrow (1)$$

$$Let\ \frac{2s^2 - 3s - 4}{(s-1)^2(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} \quad \rightarrow (2)$$

$$\therefore\ 2s^2 - 3s - 4 = A(s-1)(s+1) + B(s+1) + C(s-1)^2 \ \ \rightarrow (3)$$

$Put\ s = 1,$

$$\therefore 2 - 3 - 4 = 0 + B(2) + 0$$

$$\therefore\ -5 = 2B$$

$$\therefore B = \frac{-5}{2}$$

$Put\ s = -1\ in\ (3)$

$$\therefore 2(-1) - 3(-1) - 4 = 0 + 0 + C(-2)^{\,2}$$

$$\therefore 1 = 4C$$

$$\therefore C = \frac{1}{4}$$

$Put\ s = 0\ in\ (3)$

$$\therefore 0 - 0 - 4 = A(-1)(1) - \frac{5}{2}(1) + \frac{1}{4}(-1)^2$$

$$\therefore\ -4 + \frac{5}{2} - \frac{1}{4} = -A$$

$\therefore A = \frac{7}{4}$

$\therefore$ *From* (1)*and* (2),

$$y = L^{-1}\left[\frac{7/4}{s-1} - \frac{5/2}{(s-1)^2} + \frac{1/4}{s+1}\right]$$

$$= \frac{7}{4}L^{-1}\left[\frac{1}{s-1}\right] - \frac{5}{2}L^{-1}\left[\frac{1}{(s-1)^2}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{s+1}\right]$$

$$= \frac{7}{4}e^{1t} - \frac{5}{2}e^{1t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4}e^{-1t} \quad (First\ Shifting\ Method)$$

$$= \frac{1}{4}[7e^t - 10e^t.t + e^{-t}]$$

$\therefore y = \frac{1}{4}[e^t(7-10t) + e^{-t}]$, *is the solution of the given Differential Equation.*

---

5. a) Obtain all possible Taylor's and Laurent series which represent the function $f(z) = \frac{z}{z^2 - 5z + 6}$ indicating the region of convergence. **(6)**

Solution : $f(z) = \frac{z}{z^2 - 5z + 6}$

$$f(z) = \frac{z}{(z-2)(z-3)}$$

$$= \frac{3}{z-3} - \frac{2}{z-2} \quad (By\ Partial\ Fraction)$$

Taylor's Series:

Given $a = 1$

From (1), $f(z) = 3(z-3)^{-1} - 2(z-2)^{-1}$

$$\therefore f(a) = f(1) = 3(z-3)^{-1} - 2(z-2)^{-1} = \frac{1}{2}$$

$$\therefore f'(z) = -3(z-3)^{-2} + 2(z-2)^{-2}$$

$$\therefore f'(a) = f'(1) = -3(1-3)^{-2} + 2(1-2)^{-2} = \frac{5}{4}$$

$$\therefore f''(z) = 6(z-3)^{-3} - 4(z-2)^{-3}$$

$$\therefore f''(a) = f''(1) = 6(1-3)^{-3} - 4(1-2)^{-3} = \frac{13}{4}$$

$$\therefore f'''(z) = -18(z-3)^{-4} + 12(z-2)^{-4}$$

$$\therefore f'''(a) = f'''(1) = -18(1-3)^{-4} + 12(1-2)^{-4} = \frac{87}{8}$$

$By\ Taylor's\ Series,$

$$f(z) = f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + \cdots$$

$$\therefore \frac{z}{(z-2)(z-3)} = \frac{1}{2} + (z-1).\frac{5/4}{1!} + (z-1)^2.\frac{13/4}{2!} + (z-1)^3.\frac{87/8}{3!} + \cdots$$

$$\therefore \frac{z}{(z-2)(z-3)} = \frac{1}{2} + \frac{5}{4}(z-1) + \frac{13}{8}(z-1)^2 + \frac{29}{16}(z-1)^3 + \cdots$$

$Laurent's\ series\ expansion:$

$$f(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

$Put\ u = z - 1$

$$\therefore z = u + 1$$

$$f(z) = \frac{3}{(u+1)-3} - \frac{2}{(u+1)-2}$$

$$\therefore f(z) = \frac{3}{u-2} - \frac{2}{u-1} \quad \rightarrow (1)$$

We consider following three cases

$\boldsymbol{Case\ 1; For\ |u| < 1,}$

$Obviously\ |u| < 2$

$\therefore\ |u| < 1\ and\ \left|\frac{u}{2}\right| < 1$

$$\therefore f(z) = \frac{3}{2(u/2-1)} - \frac{2}{-(1-u)}$$

$$= \frac{-3}{2}\left(1-\frac{u}{2}\right)^{-1} + 2(1-u)^{-1}$$

$$= \frac{-3}{2}\left(1+\frac{u}{2}+\frac{u^2}{2^2}+..\right) + 2(1+u+u^2+\cdots)$$

$$= -3\left(\frac{1}{2}+\frac{u}{2^2}+\frac{u^2}{2^3}+\cdots\right) + 2(1+u+u^2+\cdots)$$

$$= -3\left[\frac{1}{2}+\frac{(z-1)}{2^2}+\frac{(z-1)^2}{2^3}+\cdots\right] + 2[1+(z-1)+(z-1)^2+\cdots]$$

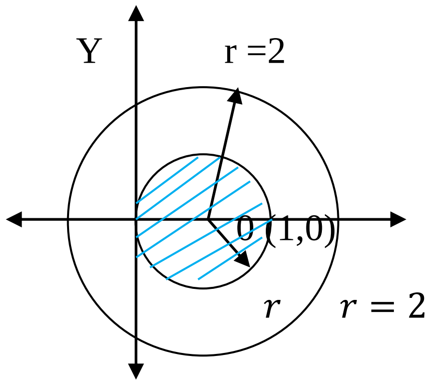
$Region\ of\ Convergence:$

$Above\ series\ is\ convergent$

$for\ \left|\frac{u}{2}\right| < 1\ and\ |u| < 1$

$i.e. |u| < 2\ and\ |u| < 1$

$i.e. |u| < 1$



$i.e. |z-1| < 1, which\ is\ the\ interior\ of\ the$

$interior\ of\ the\ circle\ with\ centre\ (1,0)\ and\ radius\ 1.$

$\boldsymbol{Case\ 2: For\ 1 < |u| < 2}$

$\therefore\ 1 < |u|\ and\ |u| < 2$

$\therefore \left|\frac{1}{u}\right| < 1\ and\ \left|\frac{u}{2}\right| < 1$

$$\therefore From\ (1), \qquad f(z) = \frac{3}{2\left({}^{u}/_{2}-1\right)} - \frac{2}{u\left(1-{}^{1}/_{u}\right)}$$

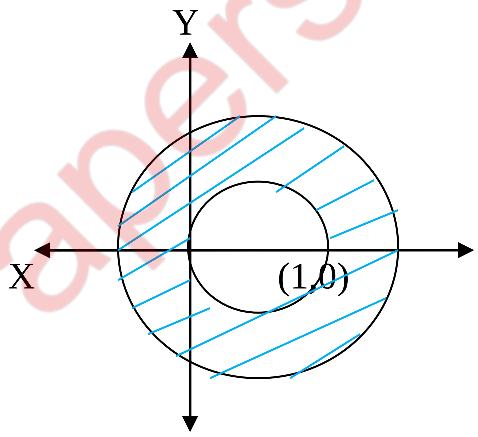
$$= \frac{-3}{2}\left(1-\frac{u}{2}\right)^{-1} - \frac{2}{u}\left(1-\frac{1}{u}\right)^{-1}$$

$$= \frac{-3}{2}\left(1+\frac{u}{2}+\frac{u^2}{2^2}+\cdots\right) - \frac{2}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right)$$

$$= -3\left(\frac{1}{2}+\frac{u}{2^2}+\frac{u^2}{2^3}+\cdots\right) - 2\left(\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right)$$

$$= -3\left[\frac{1}{2}+\frac{(z-1)}{2^2}+\frac{(z-1)^2}{2^3}+\cdots\right] - 2\left[\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+..\right]$$

Region of Convergence:



Above series is convergent

for $\left|\frac{1}{u}\right| < 1$ & $\left|\frac{u}{2}\right| < 1$

i.e. $1 < |u|$ *and* $|u| < 2$

i.e. $1 < |u| < 2$

i.e. $1 < |z-1| < 2$. *which is*

the annular region between the concentric circles with centre (1,0)

and radii 1 & 2.

***Case* 3: *For* $|u| > 2$,**

Obviously, $|u| > 1$

$\therefore 1 < |u|$ *and* $2 < |u|$

$\therefore \left|\frac{1}{u}\right| < 1$ *and* $\left|\frac{2}{u}\right| < 1$

$\therefore$ *From* (1), $f(z) = \frac{3}{u(1-{2}/{u})} - \frac{2}{u(1-{1}/{u})}$

$$= \frac{3}{u}\left(1-\frac{2}{u}\right)^{-1} - \frac{2}{u}\left(1-\frac{1}{u}\right)^{-1}$$

$$= \frac{3}{u}\left(1+\frac{2}{u}+\frac{2^2}{u^2}+\cdots\right)-\frac{2}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right)$$

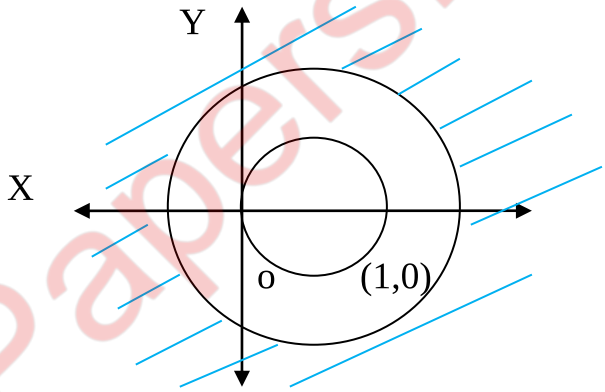
$$= 3\left(\frac{1}{u}+\frac{2}{u^2}+\frac{2^2}{u^3}+\cdots\right)-2\left(\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right)$$

$$= 3\left[\frac{1}{(z-1)}+\frac{2}{(z-1)^2}+\frac{2^2}{(z-1)^3}+\cdots\right]$$
$$-2\left[\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+..\right]$$

Region of Convergence:

Above series is convergent

for $\left|\frac{1}{u}\right|<1$ *and* $\left|\frac{2}{u}\right|<1$



$\therefore 1<|u|$ *and* $2<|u|$

$\therefore\ 2<|u|$ *and* $|u|>2$

i.e. $|z-1|>2$, *which is the exterior region of the circle with centre* $(1,0)$ *and radius* 2.

---

b) Evaluate $\displaystyle\int_0^\infty te^t\cos^2 t\,dt$ **(6)**

Solution : Consider, $L[\cos^2 t]=L\left[\frac{1+\cos 2t}{2}\right]$

$$=\frac{1}{2}\{L[1]+L[\cos 2t]\}$$

$$=\frac{1}{2}\left\{\frac{1}{s}+\frac{s}{s^2+s^2}\right\}$$

$$=\frac{1}{2}\left\{\frac{(s^2+4)+s^2}{s(s^2+4)}\right\}$$

$$=\frac{1}{2}\times\frac{2s^2+4}{s(s^2+4)}$$

$$= \frac{s^2 + 2}{s^3 + 4s}$$

$$\therefore\ L\,[t\cos^2 t] = (-1)^1 \frac{d}{ds}\left[\frac{s^2+2}{(s^3+4s)}\right] (Multiplication\ by\ 't')$$

$$\therefore \int_0^\infty e^{-st} t\cos^2 t\,dt = -\left[\frac{(s^3+4s).2s - (s^2+2).(3s^2+4)}{(s^3+4s)^2}\right]$$

$Put\ s = -1,$

$$\therefore \int_0^\infty e^{-st} t\cos^2 t\,dt = -\left[\frac{(-1-4).(-2) - (1+2).(3+4)}{(-1-4)^2}\right]$$

$$\therefore \int_0^\infty e^{-st} t\cos^2 t\,dt = \frac{11}{25}$$

----

c) *Obtain half range Fourier cosine series of*

$\boldsymbol{f(x) = x(\pi - x), 0 < x < \pi}$. ***Using Parseval's identity, deduce that***

$\boldsymbol{that - \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots}$ **(8)**

$Solution : f(x) = x(\pi - x) = \pi x - x^2$

$For\ half\ range\ cosine\ series, b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)dx$$

$$= \frac{2}{\pi}\left[\pi.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(\pi.\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - (0-0)\right]$$

$$= \frac{2}{\pi}.\frac{\pi^3}{6}$$

$$\therefore \; a_0 = \frac{x^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{\pi}\int_0^l (\pi x - x^2)\cos\frac{n\pi x}{\pi}dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2).\frac{\sin nx}{n} - \; (\pi - 2x).\frac{-\cos nx}{n^2} + \; (0-2).\frac{-\sin nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left\{\left[0 + (\pi - 2\pi).\frac{\cos nx}{n^2} + 2.\frac{\sin nx}{n^3}\right] - \left[0 + (\pi - 0).\frac{\cos 0}{n^2} + 2.\frac{\sin 0}{n^3}\right]\right\}$$

$$= \frac{2}{\pi}\left[0 - \pi.\frac{(-1)^n}{n^2} + 0 - 0 - \pi.\frac{1}{n^2} - 0\right]$$

$$= \frac{2}{\pi} \times \frac{-\pi}{n^2}[(-1)^n + 1]$$

$$\therefore \; a_n = \frac{-2}{n^2}[(-1)^n + 1]$$

$\therefore$ *Half range Fourier cosine series is,*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\therefore x(\pi - x) = \frac{\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{-2}{n^2}[(-1)^n + 1]\cos\frac{n\pi x}{\pi}$$

$$= \frac{\pi^2}{6} - 2\left[\frac{2\cos 2x}{2^2} + 0 + \frac{2\cos 4x}{4^2} + 0 + \frac{2\cos 6x}{6^2} + \cdots\right]$$

$$= \frac{\pi^2}{6} - 2 \times \frac{2}{2^2}\left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right]$$

$$\therefore x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right] \rightarrow (1)$$

Deduction 1: *Put* $x = 0$ *in* (1)

$$\therefore 0 = \frac{\pi^2}{6} - \left[\frac{\cos 0}{1^2} + \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} + \cdots\right]$$

$$\therefore 0 - \frac{\pi^2}{6} = -\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{1}^{\infty} \frac{1}{n^2}$$

Deduction 2: *Put* $x = \frac{\pi}{2}$ *in* (1)

$$\therefore \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - \left[\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \cdots\right]$$

$$\therefore \frac{\pi}{2} \cdot \frac{\pi}{2} - \frac{\pi^2}{6} = -\left[\frac{-1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Deduction 3: *Using Parseval's Identity,*

$$\frac{1}{l}\int_0^l [f(x)]^2\, dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\therefore \frac{1}{\pi}\int_0^{\pi} (\pi x - x^2)^2\, dx = \left(\frac{\pi^2}{6}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} \left\{\frac{-2}{n^2}[(-1)^n + 1]\right\}^2 + 0$$

$$\therefore \frac{1}{\pi}\int_0^{\pi} (\pi^2 x^2 - 2\pi x^3 + x^4)\, dx = \frac{\pi^4}{36} + \frac{1}{2}\left[0 + \frac{16}{2^4} + 0 + \frac{16}{4^4} 0 + \frac{16}{6^4} + \cdots\right]$$

$$\therefore \frac{1}{\pi}\left[\frac{\pi^2 x^3}{3} - \frac{2\pi x^4}{4} + \frac{\pi^5}{5}\right]_0^{\pi} = \frac{\pi^4}{36} + \frac{1}{2} \times \frac{16}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + ..\right]$$

$$\therefore \frac{1}{\pi}\left[\left(\frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5}\right) - (0 - 0 + 0)\right] = \frac{\pi^4}{36} + \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{1}{\pi} \times \frac{\pi^5}{30} - \frac{\pi^4}{36} = \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{180} = \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{90} = \sum_{1}^{\infty}\frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

---

6. *a*) *Find the image of the circle* $|z| = 2$ *under the transformation* $w = z + 3 + 2i$. *Draw the sketch.* **(6)**

Solution: *Transformation*: $w = z + 3 + 2i$.

$\therefore w - 3 - 2i = z \quad \rightarrow (1)$

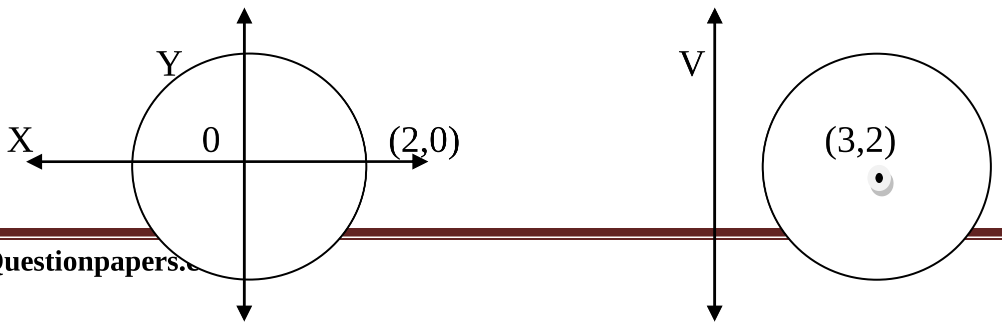
Now, given, $|z| = 2$, *which is a circle with centre* $(0,0)$ *and radius* $= 2$ *in* $z-$*plane*.

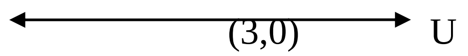
$\therefore\ |w - 3 - 2i| = 2$ (*From* 1)

$\therefore\ |u + iv - 3 - 2i| = 2$ (*we put* $w = u + iv$)

$\therefore |(u - 3) + i(v - 2)|2,$

which is a circle with centre $(3,2)$ *and radius* $= 2$ *in* $W-$*plane*.





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b) ***A rectangular metal plate with insulated surface of width l and so long as compared to its breadth that it can be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = u_0 \sin\left(\frac{\pi x}{l}\right)$ for $0 < x < l$ and other long edges $x = 0$ and $x = l$ and the short edges are kept at zero degrees temperature, find the function $u(x, y)$ describing the steady state, assuming that in the steady state the heat distribution function $u(x, y)$ satisfies the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.*** **(6)**

Solution :

We assume that in the steady state the heat distribution function $u(x, y)$ satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \to (1)$$

Its solution is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my}) \qquad \to (2)$$

The given conditions are

1) $u \to 0$ *as* $y \to \infty$ *for all* x;

2) $u = 0$ *if* $x = 0$ *for all* y;

$3)\ u = 0\ if\ x = l\ for\ all\ y;$

$4)\ u = u_0 \sin\frac{\pi x}{l}\ if\ y = 0\ for\ 0 < x < a.$

Using these initial conditions

$1)\ u \to 0\ as\ y \to \infty. Hence\ , we\ should\ have\ c_3 = 0.$

$\therefore\ u = (c_1 \cos mx + c_2 \sin mx)c_4 e^{-my} \quad \to (3)$

$2)\ u = 0\ when\ x = 0. Hence, from\ (3), we\ get,$

$0 = c_1 c_4 e^{-my} \quad \therefore c_1 = 0 \quad \therefore u = c_2 c_4 \sin mx\ e^{-my}$

$i.e. u = c_5 \sin mx\ e^{-my} \quad \to (4)$

$3)\ u = 0\ when\ x = a. Hence, from\ (4), we\ get\ 0 = c_5 \sin ml\ e^{-my}$

$\therefore ma = n\pi \quad \therefore m = \frac{n\pi}{a}, \qquad n = 1,2,3, \dots \quad \to (5)$

$\therefore\ u = c_5\ \sin\frac{n\pi x}{l}. e^{-n\pi y/l} \quad \to (6)$

Hence, the general solution is

$u = \sum b_n \sin\frac{n\pi x}{l}. e^{-n\pi y/l}\ \ where\ n = 1,2,3, \dots \quad \to (7)$

$4)\ But, u = u_0 \sin\left(\frac{\pi x}{l}\right)\ if\ y = 0.$

$Hence, from\ (7)we\ get\ ,\ \ u_0 \sin\left(\frac{\pi x}{l}\right) = \sum b_n \sin\frac{n\pi x}{l}.$

$Hence, for\ n = 1, \qquad b_1 \sin\left(\frac{\pi x}{l}\right) = u_0 \sin\left(\frac{\pi x}{l}\right) \quad \therefore b_1 = u_0\ \ and$

$b_2 = b_3 = \cdots .. =$

$Putting\ these\ values\ of\ b_n, we\ get, from\ (7)the\ solution\ of\ (1)as$

$u = u_0 \sin\left(\frac{\pi x}{l}\right) e^{-\pi y/l}\ .$

$\boldsymbol{c)\ Production\ (in\ metric\ kiloton\ )of\ when\ in\ a\ country\ is\ given\ by}$

$\boldsymbol{the\ following\ data,}$

Year (x)	2005	2007	2009	2011	2013	2015	2017
Production(y)	8	12	15	19	21	22	25

Fit a straight line to the data and estimate the production in the year 2010. **(8)**

Solution :

x	y	$X = \dfrac{x - 2011}{2}$	Y=y	X^2	XY
2005	8	-3	8	9	-24
2007	12	-2	12	4	-24
2009	15	-1	15	1	-15
2011	19	0	19	0	0
2013	21	1	21	1	21
2015	22	2	22	4	44
2017	25	3	25	9	75
	TOTAL	0	122	28	77

$Here\ , n = 7$

$Let\ the\ equation\ of\ straight\ line\ by\ Y = a + bX \quad \rightarrow (1)$

$Subject\ to,$

$$\sum Y = an + b \sum X$$

$\therefore 122 = 7a + 0$

$\therefore a = 17.4286$

$$And, \sum XY = a \sum X + b \sum X^2$$

$\therefore 77 = 0 + 28b$

$\therefore b = 2.75$

$\therefore From\ (1), the\ equation\ of\ straight\ line\ is$

$Y = 17.4286 + 2.75\,X$

$Re - substitute\ \ X\ and\ Y.$

$$y = 17.4286 + 2.75\left(\frac{x - 2011}{2}\right)$$

$$\therefore y = 17.4286 + 1.375\ (x - 2011)$$

$When\ x = 2010$

$$y = 17.4286 + 1.375(2010 - 2011) = 16.0536$$

$Hence,$

$Equation\ of\ straight\ line\ fit\ is$

$y = 17.4286 + 1.375(x - 2011)$ $and\ Production\ in\ the\ year$ 2010 in

16.0536 $metric\ kiloton.$
