MECHANICAL ENGINEERING

APPLIED MATHEMATICS III

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Q1 a) Find Laplace transform of f(t) = e^{-4t} sin3t.cos2t. (5)

sin2t.sin3t = $\frac{1}{2}.2\sin3t\sin2t = \frac{-1}{2}[\cos5t - \cos t]$

$\therefore$L(sin2t.sin3t) = $\frac{-1}{2}[L(\cos5t) - L(\cos t)] = \frac{-1}{2}\left[\frac{s}{s^2+5^2} - \frac{s}{s^2+1^2}\right]$

Now, by shifting theorem,

$$L(e^{-4t}\sin2t.\sin3t) = \frac{-1}{2}\left[\frac{s+4}{(s+4)^2+5^2} - \frac{s+4}{(s+4)^2+1^2}\right]$$

$$= \frac{-1}{2}\left[\frac{s+4}{s^2+8s+41} - \frac{s+4}{s^2+8s+17}\right]$$

$$= \frac{\mathbf{12(s+4)}}{\mathbf{(s^2+8s+41)(s^2+8s+17)}}$$

Q1 b) Show that the set of functions f(x) = 1, g(x) =x are orthogonal on (-1,1). Determine the constants a and b such that the function h(x) = 1+ax+bx^2 is orthogonal to both f(x) and g(x). (5)

$\int_{-1}^{1} f(x).g(x)dx = \int_{-1}^{1}(x)dx = 0$

Hence, the given sets are orthogonal on each other over (-1,1).

$\int_{-1}^{1} f(x).h(x)dx = 0 \quad \therefore \int_{-1}^{1} 1.(-1+ax+bx^2)dx = 0$

$\therefore [-x + \frac{ax^2}{2} + \frac{bx^3}{3}] = 0 \qquad \therefore -2 + \frac{2b}{3} = 0 \qquad \therefore b = 3$

$\int_{-1}^{1} g(x).h(x)dx = 0 \quad \therefore \int_{-1}^{1} x.(-1+ax+bx^2)dx = 0$

$\therefore [\frac{-x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4}] = 0 \qquad \therefore a = 0$

$\therefore h(x) = 3x^2 - 1$

Q1 c) Evaluate $\int(z^2 - 2\bar{z} + 1)\,dz$ where C is circle |z| = 1. (5)

Let $z = re^{i\theta}$ $\quad \therefore \bar{z} = re^{-i\theta}$ $\qquad$ Here, r = 1

Putting the values in the equation,

$\int (z^2 - 2\bar{z} + 1)\, dz \quad = \quad \int (e^{i\theta})^2 - 2\, e^{-i\theta} + 1)\, ie^{i\theta} d\theta$
θ goes from 0 to 2π

$$= \int_0^{2\pi} (e^{2i\theta} - 2\, e^{-i\theta} + 1)\, ie^{i\theta} d\theta$$

$$= i\int_0^{2\pi} (e^{3i\theta} - 2\, e^{0} + e^{i\theta}) d\theta$$

$$= i[\frac{e^{3i\theta}}{3i} - 2\,\theta + \frac{e^{i\theta}}{i}]$$

$$= [\frac{e^{6i\pi}}{3} - 4i\pi + e^{i2\pi}] - [\frac{1}{3} + 1]$$

$= \quad [\quad \frac{1}{3}(\cos 6\pi + i \sin 6\pi) - 4i\pi + (\cos 2\pi +$
$i\sin 2\pi)] - \frac{4}{3}$

$\therefore \int (\mathbf{z^2 - 2\bar{z} + 1})\, \mathbf{dz = - 4i\pi}$

Q1 d) Compute the Spearman's Rank correlation coefficient R and Karl Pearson's correlation r from the following data, (5)

x	**12**	**17**	**22**	**27**	**32**
y	**113**	**119**	**117**	**115**	**121**

Sr.No	X	X	x^2	Y	y	y^2	xy	R_1	R_2	$(R1-R2)^2$
1	12	-10	100	117	-4	16	40	5	5	0
2	17	-5	25	119	2	4	-10	4	4	4
3	22	0	0	117	0	0	0	3	3	0
4	27	5	25	115	-2	4	-10	2	2	4
5	32	10	100	121	4	16	40	1	1	0
N = 5	110			585		40	60			8

$R = 1 - \frac{6\sum D^2}{N^3 - N}$ **= 0.6** $\qquad \bar{X} = \frac{110}{5} = 22\,,\ \bar{Y} = \frac{585}{5} = 117$

$r = \frac{\sum xy}{\sqrt{\sum x^2 . \sum y^2}}$ **= 0.6**

2.a) Using Laplace transform, evaluate $\int_0^{\infty} e^{-t} \int_0^t \frac{\sin u}{u}\, du\, dt$ (6)

L (sin u) = $\frac{1}{(s^2+1)}$

L $(\frac{\sin u}{u}) = \int_s^{\infty} \frac{ds}{(s^2+1)} = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$

$\therefore L(\int_0^t \frac{\sin u}{u}\, du) = \frac{1}{s} \cot^{-1}s$

Now, $\int_0^{\infty} e^{-st} \int_0^t \frac{\sin u}{u}\, du\, dt = \frac{1}{s} \cot^{-1}s$

Put s =1,

$\therefore \int_0^{\infty} e^{-t} \int_0^t \frac{\sin u}{u}\, du\, dt = \cot^{-1}s = \frac{\pi}{4}$

2.b) Find an analytic function f(z) = u +iv, if

$u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\sin y\}$ (6)

f(z) = u +iv

$u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\sin y\}$

$u_x = \cos y[e^{-x}(2x) + (x^2 - y^2)e^{-x}] + \sin y[e^{-x}(2y) + 2xy\, e^{-x}]$

$u_y = e^{-x}[(x^2 - y^2)(-\sin y) - \cos y(-2y)] + e^{-x}[2xy\cos y - \sin y\, 2x]$

$f'(z) = u_x - iu_y$

$= \cos y[e^{-x}(2x) + (x^2 - y^2)e^{-x}] + \sin y[e^{-x}(2y) + 2xy\, e^{-x}] - i\{e^{-x}[(x^2 - y^2)(-\sin y) - \cos y(-2y)] + e^{-x}[2xy\cos y - \sin y\, 2x]\}$

Put x=z and y= 0

$\therefore f'(z) = (2ze^{-z} + e^{-z}z^2)$

$\therefore f(z) = \int(2ze - z + e - z\, z2)dz$

$\therefore f(z) = -4ze^{-z} - 3e^{-z} - z^2e^{-z}$

2.c) Obtain Fourier series of $f(x) = x^2$ in $(0,2\pi)$. Hence, deduce that-

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots .. \quad \textbf{(8)}$$

Let c=0 and c + 2l = 2 π

∴ l = π

$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx \quad = \frac{1}{\pi}\int_0^{2\pi} x^2 dx \quad = \frac{8\pi^2}{3}$

$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos\frac{n\pi x}{\pi}dx$

Solving the above integral and applying the limits, we get

$a_n = \frac{4}{n^2}$

$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin\frac{n\pi x}{\pi}dx$

Solving the above integral and applying the limits, we get

$b_n = \frac{-4\pi}{n}$

In Fourier series,

$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$\therefore x^2 = \frac{8\pi^2/3}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos\frac{n\pi x}{\pi} + \sum_{n=1}^{\infty}\frac{-4\pi}{n}\sin\frac{n\pi x}{\pi}$

$\therefore x^2 = \frac{4\pi^2}{3} + 4[\frac{\cos 1x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots..] - 4\pi[\frac{\sin 1x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\ldots]$

Deduction: Put x = π

$\therefore \pi^2 = \frac{4\pi^2}{3} + 4[\frac{\cos 1\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \ldots\ldots..] - 4\pi[\frac{\sin 1\pi}{1} + \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} + \ldots]$

$\therefore \frac{-\pi^2}{3} = -4[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{-1}{3^2} + \ldots..] - 0$

$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots..$

3.a) Using Bender- Schmidt method, solve $\frac{\partial^2 u}{\partial x^2}$ - $\frac{\partial u}{\partial t}$ = 0, subject to the conditions, u(0,t) = 0 , u(4,t) = 0, u(x,0) = $x^2(16 - x^2)$ taking h =1, for 3 minutes. (6)

a = 1 h = 1

$k = a\frac{h^2}{2} = 0.5$

We divide x interval into 4 parts by taking h=1

We also divide time interval by taking k =0.3 up to 3

$t_0 = 0, t_1 = 0.5, t_2 = 1, t_3 = 1.5, t_4 = 2, t_5 = 2.5, t_6 = 3$

By data, u (0, t) = 0

Hence, for all x = 0 and t = 0, 0.5, 1, 1.5, 2, 2.5, 3

∴ u (0, t) = 0 for all t.

By data, u(4,t) = 0

Hence, for all x = 4 and t = 0, 0.5, 1, 1.5, 2, 2.5, 3

Now, $u(x,0) = x^2(16 - x^2)$

We now calculate u for t=0 and x= 1,2,3,4

When x= 0, t = 0, $u = 0^2(16 - 0^2) = 0$

x= 1, t = 0, $u = 1^2(16 - 1^2) = 15$

x= 2, t = 0, $u = 2^2(16 - 2^2) = 48$

x= 3, t = 0, $u = 3^2(16 - 3^2) = 63$

x= 4, t = 0, $u = 4^2(16 - 4^2) = 0$

Now, $c = \frac{a+b}{2}$

t,x	0	1	2	3	4
0	0	15	48	63	0
0.5	0	24	39	24	0
1	0	19.5	24	19.5	0
1.5	0	12	19.5	12	0

2	0	9.75	6	9.75	0
2.5	0	3	9.75	3	0
3	0	4.875	3	4.875	0

3.b) Using convolution theorem, find the inverse Laplace transform of

$$\mathbf{F(s)} = \frac{s^2+s}{(s^2+1)(s^2+2s+2)} \qquad \textbf{(6)}$$

Let $\varphi_1 = \frac{s+1}{(s^2+2s+2)}$ and $\varphi_2 = \frac{s}{(s^2+1)}$

$L^{-1}\frac{s+1}{(s+1)^2+1} = e^{-u}$ $L^{-1}\frac{s}{(s^2+1)} = e^{-u}\cos u$, $\quad L^{-1}\frac{s}{(s^2+1)} = \cos u$

$\therefore L^{-1}\left[\frac{s+1}{(s^2+2s+2)} \cdot \frac{s}{(s^2+1)}\right] = \int_0^t e^{-u}\cos u \, . \cos(t-u)\, du$

$$= \frac{1}{2}\int_0^t e^{-u}[\cos(2u-1) + \cos t]\, du$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-u}[-\cos(2u-t) + 2\sin(2u-t) - e^{-u}\cos t\right]$$

$$= \frac{\mathbf{1}}{\mathbf{10}}\left[\mathbf{e^{-t}}[(\mathbf{2\sin t - 6\cos t}) + (\mathbf{2sint + 6cost})\,\right]$$

3.c) Using Residue theorem, evaluate

i) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ **ii)** $\oint \frac{z^2}{(z+1)^2(z-2)}$ **dz , C: |z| = 1.5** **(8)**

i) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

$e^{i\theta} = z \qquad i\, e^{i\theta} d\theta = dz \qquad \therefore \frac{dz}{iz} = d\theta$ and $\cos\theta = \frac{z + z^{-1}}{2}$

$\therefore I = \oint \frac{1}{\left[2+\frac{z^2+1}{2z}\right]} \cdot \frac{dz}{iz} \quad = \frac{4}{i}\oint \frac{zdz}{[\,z^2+4z+1]}$ where C is the circle $|z| = 1.5$

Now the poles of f(z) are given by $z^2 + 4z + 1 = 0$

$\therefore z = -2 \pm \sqrt{3}$

Both the poles are of order 2. But the pole $\alpha = -2-\sqrt{3}$ lies outside the circle and the pole $\beta = -2+\sqrt{3}$ lies inside the circle.

$\therefore$ Residue at (z = β) = $\lim_{z\to\beta}\frac{1}{1!}\frac{d}{dz}\left[(z-\beta)^2\cdot\frac{z}{(z-\beta)^2(z-\alpha)^2}\right]$

Solving the above limit, we get

$$=-\frac{\beta+\alpha}{(\beta-\alpha)^3}$$

But, $\beta+\alpha = -4$ and $\beta-\alpha = 2\sqrt{3}$

$\therefore$ Residue (at z = β) = $\frac{4}{24\sqrt{3}} = \frac{1}{6\sqrt{3}}$

$\therefore$ I = $2\pi i\cdot\frac{4}{i}\cdot\frac{1}{6\sqrt{3}} = \boldsymbol{\frac{4\pi}{3\sqrt{3}}}$

ii) $\oint\frac{z^2}{(z+1)^2(z-2)}dz$, C: |z| = 1.5

Now, the poles are given by $(z+1)^2(z-2) = 0$

$\therefore$ z = -1, 2

The pole z = -1 lies inside the circle |z| = 1.5 and the pole z = 2 lies outside.

Now, the residue of f(z) (at z = -1) = $\lim_{z\to-1}\frac{1}{1!}\frac{d}{dz}\left[(z+1)^2\cdot\frac{z^2}{(z+1)^2(z-2)^2}\right]$

$$= \lim_{z\to-1}\frac{d}{dz}\left[\frac{z^2}{(z-2)^2}\right]$$

$$= \lim_{z\to-1}\left[\frac{(z-2)^2.2z-z^2.2\,(z-2)}{(z-2)^4}\right] = \lim_{z\to-1}\left[\frac{-4z}{(z-2)^3}\right] = \frac{-4}{27}$$

$\therefore$ I = $2\pi i.\frac{-4}{27} = \boldsymbol{\frac{-8\pi i}{27}}$

4.a) Solve by Crank Nicholson simplified formula $\frac{\partial^2u}{\partial x^2}$ -16$\frac{\partial u}{\partial t}$ = 0, u(0,t) = 0, u(1,t) – 200t, u(x,0) = 0 taking h = 0.25 for one time step. (6)

a = 16 and h = 0.25

$k = ah^2 = 16 \text{ x } \frac{1}{16} = 1$

Now, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

By data,

When t=0, u=0, x = 0, 0.25, 0.5, 0.75, 1.00

When x =0, for all values of t, u=0 when t =0,1

When x =1 and t =0, u =0 and when x =1, t=1, u =200 x 1 =200.

t,x	0.00	0.25	0.50	0.75	1.00
0	0	0	0	0	0
1	0	u_1	u_2	u_3	200

$e = \frac{1}{4}$ (a+b+c+d)

$u_1 = \frac{1}{4}(0+0+0+u_2) = u_2/4$

$u_2 = \frac{1}{4}(0+0+u_1+u_3) =u_1+ u_3/4$

$u_3 = \frac{1}{4}(0+0+u_2+u_4) =u_2+ 200/4$

$\therefore u_2 = \frac{1}{4}(u_2/4 + u_2+ 200/4)$

$\therefore u_2 = 100/7$

$u_1 = u_2/4 = 25/7$

$u_3 = u_2+ 200/4 =375/7$

The final table is:

t,x	0.00	0.25	0.50	0.75	1.00
0	0	0	0	0	0
1	0	3.5714	14.2857	53.5714	200

4.b) Obtain the Laurent series which represent the function

$f(z) = \frac{4z+3}{z(z-3)(z+2)}$ in the regions, i) $2 < |z| < 3$ ii) $|z| > 3$ (6)

$f(z) = \frac{a}{z} + \frac{b}{z-3} + \frac{c}{z+2}$

$\therefore$ 4z +3 = a(z-3)(z-2) + b z(z+2) +c z(z-3)

When z =0,   3 = -6a $\therefore$ a =-1/2

When z = -2,   -5 = 10c $\therefore$ c = -1/2

When z = 3,   15 = 15b $\therefore$ b= 1

$\therefore \frac{\mathbf{4z+3}}{\mathbf{z(z-3)(z+2)}} = \frac{-1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)}$

When **2 < |z| < 3,**

$f(z) = \frac{-1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)}$

$f(z)= \frac{-1}{2z} - \frac{1}{3(1-\frac{z}{3})} - \frac{1}{2z(1+2/z)}$

$= \frac{-1}{2z} - \frac{1}{3}(1 + \frac{z}{3} + \frac{z^2}{3} + \cdots) - \frac{1}{2z}(1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots\ldots.)$

When **|z| >3**

$f(z) = \frac{-1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)}$

$f(z)= \frac{-1}{2z} - \frac{1}{z(1-\frac{3}{z})} - \frac{1}{2z(1+2/z)}$

$= \frac{-1}{2z} - \frac{1}{z}(1 +\frac{3}{z} + \frac{3^2}{z^2} \ldots\ldots.) - \frac{1}{2z}(1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots\ldots.)$

4.c) Solve (D^2 – 3D +2) y = $4e^{2i}$ with y (0) = -3 and y' (0) = 5 where D = $\frac{d}{dt}$ (8)

L(y) = $\bar{y}$

Taking Laplace transform,

L(y'') – 3L(y') + 2L(y) = a L(**$4e^{2i}$**)

But, L(y') = s$\bar{y}$ y(0) = s$\bar{y}$ + 3

And L(y'') = $s^2\bar{y}$ - sy(0) – y'(0) = s^2y +3s – 5

$\therefore$ the equation becomes,

$s^2\bar{y} + 3s - 5 - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s-2}$

Solving both sides, we get

$\bar{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

Taking inverse Laplace transform,

$y = -7\ L^{-1}(\frac{1}{s-1}) + 4L^{-1}(\frac{1}{s-2}) + 4L^{-1}\frac{4}{(s-2)^2}$

$= -7e^{i}\ L^{-1}\frac{1}{s} + 4\ e^{2i}L^{-1}\frac{1}{s^2}$

$\mathbf{= -7e^{i} + 4e^{2i} + 4ie^{2i}}$

5.a) Find the bilinear transformation under which 1, i, -1 from the z plane are mapped onto 0,1,∞ of w-plane. (6)

Let the transformation be $w = \frac{az+b}{cz+d}$

Putting the given values of z and w, we get,

$0 = \frac{a+b}{c+d}$ $\quad 1 = \frac{ai+b}{ci+d}$ $\quad \infty = \frac{-a+b}{-c+d}$

$\therefore a + b = 0$ $\quad \therefore b = -a$

$\therefore -c + d = 0$ $\quad \therefore d=c$

Now,

$ai = b = ci+ d$ $\quad \therefore ai- a = ci +c$ $\quad \therefore ai + b = ci + d$

$\therefore ai - a = ci = c$ $\quad a(i-1) = c(i + 1)$

$\therefore c = a\frac{i-1}{i+1}$ $\quad \therefore c = a\frac{(i^2-2i+1)}{i^2-1} = ai$

$d = c = ai$ $\quad \therefore w = -i\frac{z-1}{z+1}$

Now, when $|\mathbf{w}|=1$, $|-i\frac{z-1}{z+1}| = 1$

$\therefore |z-1| = |z+1|$

$\therefore |(x-1) - iy)| = |(x+1) + iy|$

$\therefore (x-1)^2 + y^2 = (x+1)^2 + y^2$

$\therefore 4x = 0$ $\quad\quad\quad\quad$ **$\therefore x = 0$**

Hence, the map is in y axis.

5.b) Find the Laplace transform of

$f(t) = t,\ 0 < t < \pi$

$\pi - t,\ \pi < t < 2\pi$ and $f(t+2\pi) = f(t)$ **(6)**

Since, f(t) is periodic with period a = 2π, we have

$$Lf(t) = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st}.tdt + \int_{\pi}^{2\pi} e^{-st}(\pi - t)dt\right.$$

Solving the above equation, we get

$$\mathbf{Lf(t) = \frac{1-(1+\pi s)e^{-\pi s}}{s^2(1+e^{-\pi s})}}$$

5.c) Obtain half range Fourier cosine series of f(x) = x, 0 < x<2. Using Parseval's identity, deduce that –

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \quad \textbf{(8)}$$

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l}$ here l=2.

$$\therefore a_0 = \frac{1}{l}\int_0^l f(x)dx = 1$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

Solving the above equation, we get,

$$a_n = (2.(0) + \frac{\cos n\pi}{n^2\pi^2/2^2} - 0 - \frac{1}{n^2\pi^2/2^2}) = \frac{[(-1)^n - 1]}{n^2\pi^2/2^2}$$

$$\therefore x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\ldots\right]$$

By Parseval's identity,

$\frac{1}{l}\int_0^l [f(x)]^2 dx = \frac{1}{2}[2a_0^2 + a_1^2 + a_2^2 + \ldots.]$

$\therefore$ LHS $= \frac{1}{l}\int_0^l [x]^2 dx = \frac{4}{3}$

$\therefore \frac{4}{3} = \frac{1}{2}[\ 2 + \frac{64}{\pi^2}\ \{\frac{1}{1^4} + \frac{1}{3^4} + \ \frac{1}{5^4} + \ldots\ldots\ldots\}]$

$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

6.a) Using Contour Integration, evaluate:

$$\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9}\, dx \qquad \textbf{(6)}$$

$f(x) = \frac{x^2+x+2}{x^4+10x^2+9}$

$f(z) = \frac{z^2+z+2}{z^4+10z^2+9}$

For poles, $z^4+10z^2 + 9 = 0$

Solving the equation, we get

$z = \pm i$ and $z = \pm 3i$

z = i,3i are the poles which lie inside the circle.

Residue of f(z) at z =i is:

$$= \lim_{z\to i}(z-i)[\frac{z^2+z+2}{z^4+10z^2+9}]$$

$$= \lim_{z\to i}(z-i)[\frac{z^2+z+2}{(z^2+9)(z^2+1)}]$$

$$= \lim_{z\to i}(z-i)[\frac{z^2+z+2}{(z^2+9)(z+i)(\,z-i)}]$$

$$= \frac{1+i}{16i}$$

Residue of f(z) at z =3i is:

$$= \lim_{z\to i}(z-i)[\frac{z^2+z+2}{z^4+10z^2+9}]$$

$= \lim_{z\to i}(z-i)[\frac{z^2+z+2}{(z^2+9)(z^2+1)}]$

$= \lim_{z\to i}(z-i)[\frac{z^2+z+2}{(z-3i)(z+3i)(z^2+1)}]$

$=\frac{7-3i}{48i}$

Sum of residues := $\frac{5}{24i}$

$\therefore \int_{-\infty}^{\infty}\frac{x^2+x+2}{x^4+10x^2+9} = 2\pi i \, . \frac{5}{24i} = \frac{5\pi}{12}$

6.b) Using least square method, fit a parabola, y= a+bx+cx^2 to the following data, (6)

x	-2	-1	0	1	2
y	-3.150	-1.390	0.620	2.880	5.378

Sr. No	x	y	x^2	x^3	x^4	xy	x^2y
1	-2	-3.150	4	-8	16	6.300	-12.600
2	-1	-1.390	1	-1	1	1.390	-1.390
3	0	0.620	0	0	0	0.000	0.000
4	1	2.880	1	1	1	2.880	2.880
5	2	5.378	4	8	16	10.756	21.512
N = 5	0	4.338	10	0	34	21.326	10.402

The normal equations are

$\sum y = Na + b\sum x + c\sum x^2$ $\therefore$ 4.338 = 5a + 10c

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ $\therefore$ 21.325 = 10b

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4$ $\therefore$ 10.402 = 10a + 34c

Solving these three equations, we get

a = 0.621, b =2.1326, c= 0.1233

Hence, the parabola is **y = 0.621 + 2.1326x + 0.1233x^2**

6.c)Determine the solution of one dimensional heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ under the boundary conditions u(0,t) = 0, u(l,t) = 0, u(x,0) = x,(0< x < l), l being the length of the rod. (8)

The equation of heat flow $\frac{\partial u}{\partial t} = c^2 \frac{\partial u}{\partial t}$ .....(1)

Since the temperature of the two ends of the rod are zero, its solution is of the form

$$u = (c_1 \cos mx + c_2 \sin mx)e^{-m^2c^2t} \quad \ldots\ldots (2)$$

Using initial conditions,

(i) u =0 when x = 0, we get, from (2) $0 = c_1 e^{-m^2c^2t}$

$\therefore c_1 = 0$

$\therefore$the solution (2) becomes $u = c_2 \sin mx \; e^{-m^2c^2t}$ .... (3)

(ii) u =0 when x=l, we get, from (3) $0 = c_2 \sin ml \; e^{-m^2c^2t}$

ml = nπ therefore $m = \frac{n\pi}{l}$

Hence, the general solution is $u = \sum b_n \sin\frac{n\pi x}{l} e^{-n^2\pi^2c^2\frac{t}{l^2}}$ ....
(4)

(iii) When l = 0, u = x, we get from (4) $\sum b_n \sin\frac{n\pi x}{l}$ ... (5)

But this is a half-range Fourier sine series for the function f(x) = x in the range 0 to l.

$$b_n = \frac{2}{l}\int_0^l x \sin\frac{n\pi x}{l} dx = \frac{2}{l}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)(1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]$$

$$= \frac{2}{l}\left[-\frac{l^2}{n\pi}\cos n\pi\right] = -\frac{2l}{n\pi}\cos n\pi \, , n = 1,2,3 \ldots\ldots \quad \ldots\ldots(6)$$

Hence, from (4) we get solution as

$$u = \frac{2l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} e^{-\pi^2c^2\frac{t}{l^2}} - \frac{1}{2}\sin\frac{2\pi x}{l} e^{-4\pi^2c^2\frac{t}{l^2}} + \cdots \ldots\right]$$