M3 SOLUTION OF QUESTION PAPER

(CBCGS DEC 2017)

Q.1] a) Find Laplace transform of $\boldsymbol{f(t) = te^{-3t}sint}$. **(5)**

Solution:

$$L[sint] = \frac{1}{s^2 + 1^2}$$

$$\therefore L[t.sint] = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2 + 1}\right)$$

$$= -1 \times \frac{-1}{(s^2 + 1)^2} \times 2s$$

$$= \frac{2s}{(s^2 + 1)^2}$$

$$\therefore L[te^{-3t}sint] = \frac{2(s+3)}{[(s+3)^2 + 1]^2} (First\ shifting\ Method)$$

$$= \frac{2(s+3)}{[s^2 + 6s + 9 + 1]^2}$$

$$= \frac{2(s+3)}{(s^2 + 6s + 10)^2}$$

$$\therefore \boldsymbol{L[te^{-3t}sint] = \frac{2(s+3)}{(s^2 + 6s + 9)^2}}$$

---

b) Obtain Complex form of Fourier series of $\boldsymbol{f(x) = e^x, -1 < x < 1}$ in $\boldsymbol{(-1, 1)}$. **(5)**

Solution:

Let $c = -1$ and $c + 2l = 1$

$\therefore -1 + 2l = 1$

$\therefore 2l = 2$

$\therefore l = 1$

Given, $f(x) = e^x$

$$C_n = \frac{1}{2l}\int_{c}^{c+2l} f(x)e^{-inx/l}\, dx$$

$$= \frac{1}{2(1)}\int_{-1}^{1} e^x\ e^{-inx/1}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{x-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{(1-in\pi)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{(1-in\pi)x}}{1-inx}\right]_{-1}^{1}$$

$$= \frac{1}{2} \times \frac{1}{1-inx}\left[e^{(1-in\pi)} - e^{-(1-in\pi)}\right]$$

$$= \frac{1}{2(1-inx)} \times \left[e^{1}e^{-in\pi} - e^{-1}e^{1-in\pi}\right] \times \frac{(1+in\pi)}{(1+in\pi)}$$

Consider,

$$e^{\pm in\pi} = \cos n\pi \pm i\sin n\pi = (-1)^n \pm i0 = (-1)^n$$

$$\therefore C_n = \frac{(1+in\pi)}{2(1^2 - i^2n^2\pi^2)} \times [e^1(-1)^n - e^{-1}(-1)^n]$$

$$= \frac{(1+in\pi)}{2(1+n^2\pi^2)}(-1)^n \times [e^1 - e^{-1}]$$

$$= \frac{(1+in\pi)}{\cancel{2}(1+n^2\pi^2)}(-1)^n \times \cancel{2} \sin h1$$

In Complex Fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/_1}$$

$$\therefore \ e^x = \sum_{n=-\infty}^{\infty} \frac{(1+in\pi x)}{(1+n^2\pi^2)} \sin h1.\, e^{in\pi x/_1}$$

$$\therefore \ \boldsymbol{e^x = \sin h1 \sum_{n=-\infty}^{\infty} \frac{(1+in\pi x)}{(1+n^2\pi^2)}(-1)^n.\, e^{in\pi x/_1}}$$

---

c) Does there exist an analytic function whose real part is $\boldsymbol{u = k(1+\cos\theta)}$?Give justification. (5)

Solution:

$$u = k(1+\cos\theta)$$

$$Differentiating\ partially\ w.r.t.'r', \frac{\partial u}{\partial r} = 0 \ \rightarrow (1)$$

$$Again, differentiating\ partially\ w.r.t.'r', \frac{\partial^2 u}{\partial r^2} = 0 \ \rightarrow (2)$$

$$Similarly, differentiating\ 'u' partially\ w.r.t.'\theta',$$

$$\frac{\partial u}{\partial \theta} = k(0 - \sin\theta) = -k\ \sin\theta$$

$$Again, differentiating\ partially\ w.r.t.'\theta', \frac{\partial^2 u}{\partial \theta^2} = -k\cos\theta \ \rightarrow (3)$$

Consider Laplace's equation in polar form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} =$$

$= 0 + \frac{1}{r}.(0) + \frac{1}{r^2}.-k\cos\theta$ $(From\ 1{,}2\ and\ 3)$

$= \frac{-k}{r^2}\cos\theta$

$\neq 0$

$\therefore' u' does\ not\ satisfies\ Laplace's\ equation.$

$\therefore 'u' is\ not\ harmonic.$

$So, u\ is\ not\ a\ part\ of\ analytic\ function\ f(z) = u + iv.$

Hence, there does not exist an analytic function whose real part is

$\boldsymbol{u = k(1 + \cos\theta)}$**.**

---

d) The equations of lines are of regression are $3x + 2y = 26$ and $6x + y = 31$. Find 1) means of x and y, 2) coefficient of correlation between x and y. (5)

Solution:

$3x + 2y = 26$

$\therefore y = -\frac{3}{2}x + \frac{26}{2} \rightarrow (1)$

$And, 6x + y = 31$

$\therefore y = -6x + 31 \rightarrow (2)$

$Let\ b_1 = -\frac{3}{2}\ and\ b_2 = -6$

$Since\ |b_1| < |b_2|, \quad b_{yx} = b_1 = \frac{-3}{2}\ and\ b_{xy} = \frac{1}{b_2} = \frac{-1}{6} \rightarrow (3)$

$\therefore Equation\ (1) is\ regression\ equation\ of\ Y\ on\ X\ type\ and\ equation\ (2) is\ regression\ equation\ of\ X\ on\ Y\ type.$

$From\ (1) and\ (2), -\frac{3}{2}x + 13 = -6x + 31$

$\therefore 6x - \frac{3}{2}x = 31 - 13$

$\therefore \frac{9}{2}x = 18$

$\therefore x = 4$

$Substituting\ x = 4\ in\ (2), y = -6(4) + 31$

$\therefore y = 7$

$Now, r = \pm\sqrt{b_{yx} . b_{xy}}$

$= \pm\sqrt{\frac{-3}{2} \times \frac{-1}{6}}\ \ (From\ 3)$

$= \pm\frac{1}{2}$

$Since, b_{yx}\ and\ b_{xy}\ are\ both\ negative, 'r' is\ negative.$

$\therefore r = -1/2$

$Now, b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$\therefore \frac{-3}{2} = \frac{-1}{2} \times \frac{\sigma_y}{3}$

$\therefore \sigma_y = 9$

$Hence,$

$\boldsymbol{Means\ of\ x\ and\ y;\ \bar{x} = 4\ ;\ \bar{y} = 7}$

$\boldsymbol{Coefficient\ of\ Correlation\ (r)\ between\ x\ and\ y = 0.5\ \ \ \sigma_y = 9}$

Q. 2] $a)$ $Evaluate\ \int_0^\infty e^t\ \sin 2t\ \cos 3t\ dt.$ (6)

Solution:

$Conider, L[\cos 3t \sin 2t] = \frac{1}{2} L[2 \cos 3t \sin 2t]$

$= \frac{1}{2} L[\sin(3t + 2t) - \sin(3t - 2t)]$

$\therefore L[\sin 2t \,.\cos 3t] = \frac{1}{2} \{L[\sin 5t] - L[\sin t]\}$

$$\therefore \int_0^\infty e^{-st} \sin 2t \cos 3t \, dt = \frac{1}{2}\left\{\frac{5}{s^2 + 5^2} - \frac{1}{s^2 + 1^2}\right\}$$

$Put\ s = -1$

$$\therefore \int_0^\infty e^{t} \sin 2t \cos 3t \, dt = \frac{1}{2}\left\{\frac{5}{(-1)^2 + 25} - \frac{1}{(-1)^2 + 1}\right\}$$

$$\therefore \int_0^\infty \boldsymbol{e^{t} \sin 2t \cos 3t \, dt = \frac{-2}{13}}$$

---

b) Find the image of the square bonded by lines

$x = 0, x = 2, y = 0, y = 2$, in the z-plane under the transformation

$w = (1 + i)z + (2 - i)$. **(6)**

Solution:

$Consider, w = (1 + i)z + (2 - i)$

$\therefore u + iv = (1 + i)(x + iy) + (2 - i), (we\ put\ w = u + iv\ and\ z = x + iy)$

$\therefore u + iv = x + i^2y + ix + iy + 2 - i$

$\therefore u + iv = (x - y + 2) + i(x + y - 1)$

$Comparing\ real\ and\ imaginary\ part\ on\ both\ sides, u = x - y + 2 \ \rightarrow$ (1)

$and\ v = x + y - 1 \ \rightarrow (2)$

$Adding\ (1) and\ (2), u + v = 2x + 1 \rightarrow (3)$

$Subtract\ (2) from\ (1), u - v = -2y + 3 \ \rightarrow (4)$

$Now, when\ x = 0, (which\ is\ Y - axis\ in\ Z - plane)$

$From\ (3), u + v = 1\ (which\ is\ a\ line\ having\ slope = -1\ in\ W - plane)$

$When\ x = 2, (which\ is\ line\ parallel\ to\ Y - axis\ in\ Z - plane)$

$From\ (3), u + v = 2\ (2) + 1$

$\therefore u + v = 5\ (which\ is\ a\ line\ having\ slope = -1\ in\ W - plane)$

$When\ y = 0, (which\ is\ X - axis\ in\ Z - plane)$

$From\ (4), u - v = 3\ (which\ is\ a\ line\ having\ slope = 1\ in\ W - plane)$

$When\ y = 2, (which\ is\ line\ parallel\ to\ X - axis\ in\ Z - plane)$

$From\ (4), u - v = -2\ (2) + 3$

$u - v = -1\ (which\ is\ a\ line\ having\ slope = 1\ in\ W - plane)$

$Given, w = (1 + i)z + (2 - i)$

$Let\ w_1 = (1 + i)z, which\ is\ "Rotation\ and\ Magnification"$

$transformation, in\ which\ the\ shape\ is\ preserved.$

$\therefore w = w_1 + (2 - i), which\ is\ "Translation"\ \ transformation,$

$in\ whicht\ the\ shape\ is\ preserved.$

$So, the\ above\ four\ lines\ in\ w - plane\ will\ intersect\ to\ form\ a\ square.$

$\boldsymbol{Hence, the\ square\ (bonded\ by\ x = 0, x = 2, y = 0, y = 2) in\ the\ z - plane}$

$\boldsymbol{is\ mapped\ onto\ the\ square\ (bonded\ by\ u + v = 1, u + v = 5, u - v = 3,}$

$\boldsymbol{u - v = -1\ in\ the\ W - plane.}$

$c)$ ***Obtain Fourier series of*** $\boldsymbol{f(x) = |x|}$ ***in*** $\boldsymbol{(-\pi, \pi)}$. ***Hence,***

deduce that $\boldsymbol{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$ **(8)**

Solution:

$f(x) = |x|$

$\therefore f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)\ is\ even\ function.$

$\therefore\ b_n = 0$

$Here, l = \pi$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{\pi}\int_0^\pi |x|dx \quad = \frac{2}{\pi}\int_0^\pi xdx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi$$

$$= \frac{1}{\pi}[\pi^2 - 0] \quad (2\ get\ cancelled.)$$

$$= \pi$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{\pi}\int_0^\pi |x|\cos\frac{n\pi x}{\pi}dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx dx \quad (\pi \text{ get cancelled.})$$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n} - 1.\frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi.\frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

In Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\therefore\ |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^n - 1]\cos\frac{n\pi x}{\pi} + 0$$

$$\therefore\ |x| = \frac{\pi}{2} + \frac{2}{\pi}\left(\frac{-2\cos x}{1^2} + 0 - \frac{2\cos 3x}{3^2} + 0 - \frac{2\cos 5x}{5^2} + \cdots\right)$$

$$\therefore\ |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) \rightarrow \ (1)$$

Deduction: *Put* $x = 0$ *in* (1)

$$\therefore\ 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \cdots\right)$$

$$\therefore\ -\frac{\pi}{2} = -\frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\therefore\ \frac{-\pi}{2} \times \frac{\pi}{-4} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\therefore\ \boldsymbol{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

$\boldsymbol{Q.3]\ a)\ Find\ the\ inverse\ Laplace\ transform\ of}$

$$\boldsymbol{F(s) = \frac{s}{(s^2+9)(s^2+4)}}. \qquad \textbf{(6)}$$

Solution:

$$L^{-1}\left[\frac{s}{(s^2+9)(s^2+4)}\right] = L^{-1}\left[\frac{s}{s^2+4} \times \frac{1}{s^2+9}\right]$$

$$Let\ \emptyset_1(s) = \frac{s}{s^2+4}\ ;\ \emptyset_2(s) = \frac{1}{s^2+9}$$

$$\therefore\ f_1\ (t) = L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t\ and\ f_2\ \ (t) = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}\sin 3t$$

$$By\ Convolution\ theorem, L^{-1}[\emptyset_1(s)\emptyset_2(s)] = \int_0^t f_1(u)f_2(t-u)du$$

$$\therefore\ L^{-1}\left[\frac{s}{s^2+4} \times \frac{1}{s^2+9}\right] = \int_0^t \cos 2u.\frac{1}{3}\sin 3(t-u)du \times \frac{2}{2}$$

$$= \frac{1}{3} \times \frac{1}{2}\int_0^t 2\cos 2u.\sin(3t-3u)du$$

$$= \frac{1}{6}\int_0^t [\sin(2u+3t-3u) - \sin(2u-3t+3u)]\,du$$

$$= \frac{1}{6}\int_0^t [\sin(-u+3t) - \sin(5u-3t)]\,du$$

$$= \frac{1}{6}\left[\frac{-\cos(-u+3t)}{-1} + \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{6}\left\{\left[\cos(-t+3t) + \frac{\cos(5t-3t)}{5}\right] - \left[\cos(0+3t) + \frac{\cos(0-3t)}{5}\right]\right\}$$

$= \frac{1}{6}\left\{\left[\cos 2t + \frac{\cos 2t}{5}\right] - \left[\cos 3t + \frac{\cos 3t}{5}\right]\right\}$

$= \frac{1}{6}\left\{\frac{6}{5}\cos 2t - \frac{6}{5}\cos 3t\right\}$

$= \frac{1}{6} \times \frac{6}{5}\{\cos 2t - \cos 3t\}$ $hgjyfgjkhbkhk$

$\therefore \boldsymbol{L^{-1}\left[\frac{s}{s^2+4} \times \frac{1}{s^2+9}\right] = \frac{1}{5}(\cos 2t - \cos 3t)}$

$\boldsymbol{b)Solve\ \frac{\partial^2 u}{\partial x^2} - 100\frac{\partial u}{\partial t} = 0\ with\ u(0,t) = 0, u(1,t) = 0,}$

$\boldsymbol{u(x,0) = x(1-x), taking\ h = 0.1\ for\ three\ steps\ upto}$

$\boldsymbol{t = 1.5\ by\ Bender - Schmidt\ method.}$ **(6)**

Solution:

$We\ are\ given\ h = 0.1\ and\ a = 100$

$k = \frac{a}{2}h^2 \qquad \therefore k = \frac{100}{2}.(0.1)^2 = 0.5$

$Since, h = 0.1, and\ the\ x\ is\ 0\ to\ 1. We\ divide\ x\ interval\ into\ 10\ parts\ by$

$taking\ h = 0.1.$

$We\ also\ divide\ the\ time\ interval\ into\ 3\ parts\ by\ taking\ k = 0.5$

$t_0 = 0, t_1 = 0.5, t_2 = 1.0, t_3 = 1.5$

$By\ data, \qquad u(0,t) = 0$

$Hence, for\ all\ x = 0\ and\ t = 0, 0.5, 1.0, 1.5.$

$u(0,t) = 0\ for\ all\ t.$

$\therefore\ By\ data \qquad u(1,t) = 0$

$Hence\ for\ all\ x = 1\ and\ t = 0, 0.5, 1.0, 1.5$

$\therefore u(1,t) = 0\ for\ all\ t.$

$Now\ u(x,0) = x(1-x)$

$We\ now\ calculate\ u\ for\ t = 0\ and$

$x = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$

$\therefore when\ x = 0, t = 0, \quad u = 0(1-0) = 0$

$when\ x = 0.1, t = 0, \ u = 0.1(1-0.1) = 0.09$

$when\ x = 0.2, t = 0, \ u = 0.2(1-0.2) = 0.16$

$when\ x = 0.3, t = 0, \ u = 0.3(1-0.3) = 0.21$

$when\ x = 0.4, t = 0, \ u = 0.4(1-0.4) = 0.24$

$when\ x = 0.5, t = 0, \ u = 0.5(1-0.5) = 0.25$

$when\ x = 0.6, t = 0, \ u = 0.6(1-0.6) = 0.24$

$when\ x = 0.7, t = 0, \ u = 0.7(1-0.7) = 0.21$

$when\ x = 0.8, t = 0, \ u = 0.8(1-0.8) = 0.16$

$when\ x = 0.9, t = 0, \ u = 0.9(1-0.9) = 0.09$

$when\ x = 1.0, t = 0, \ u = 1.0(1-1.0) = 0$

t \ x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0
1	0	0.08	0.15	0.20	0.23	0.24	0.23	0.20	0.15	0.08	0
2	0	0.075	0.14	0.19	0.22	0.23	0.22	0.19	0.14	0.075	0
3	0	0.07	0.1325	0.18	0.21	0.22	0.21	0.18	0.1325	0.07	0

$\boldsymbol{Used\ Bender\ Schmidt\ formula\ c = \frac{a+b}{2}\ for\ remaining\ values.}$

$c)$ ***Using Residue Theorem evaluate*** **(8)**

$$1]\int_0^{2\pi}\frac{d\theta}{5+4cos\theta}$$

Solution:

$$Let\ z=e^{i\theta}\quad \therefore dz=ie^{i\theta}.d\theta=izd\theta\quad \therefore d\theta=\frac{dz}{iz};\ \cos\theta=\frac{z^2+1}{2z}$$

$$\therefore I=\int_c \frac{z}{5+4\left(\frac{z^2+1}{2z}\right)}.\frac{dz}{iz}$$

$$I=\int_c \frac{z}{i(2z^2+5z+2)}.\frac{dz}{z}$$

$Where\ C\ is\ the\ circle\ |z|=1$

$Now, the\ poles\ are\ given\ by\ (2z+1)(z+2)=0$

$$\therefore z=-\frac{1}{2},\ \ z=-2$$

$The\ pole\ z=-\frac{1}{2}\ lies\ inside\ the\ unit\ circle\ and\ the\ pole\ z=-2\ lies$

$outside.$

$Now, Residue\ of\ f(z)\left(at\ z=-\frac{1}{2}\right)$

$$=\lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right).\frac{z}{2[z+(1/2)](z+2)i}$$

$$=\frac{2}{2[-(1/2)+2]i}=\frac{2}{3i}$$

$$=2\pi i\left(\frac{2}{3i}\right)=\frac{4\pi}{3}$$

$$\therefore\ \int_0^{2\pi}\frac{d\theta}{5+4cos\theta}=\frac{4\pi}{3}$$

$\boldsymbol{Q.4]\ a)\ Solve\ by\ Crank - Nicholson\ simplified\ formula}$

$\boldsymbol{\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, u(0,t) = 0, u(5,t) = 100, u(x,0) = 20, taking\ h = 1\ for}$

$\boldsymbol{one - time\ step.}$

Solution:

$Here\ we\ have\ a = 1, h = 1$

$To\ use\ Crank - Nicholson\ formula, we\ must\ have$

$k = ah^2 = 1 \times 1^2 = 1$

$The\ interval\ of\ x\ is\ 0\ to\ 5. The\ subinterval\ is\ of\ size\ h = 1.$

$\therefore x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$

$By\ data\ u(0,t) = 0$

$When\ x = 0\ for\ all\ values\ of\ t, u = 0, when\ t = 0, 1.$

$By\ date\ u(x,o) = 20$

$For\ all\ values\ of\ x, when\ t = 0, x = 0, 1, 2, 3,4, 5.$

$By\ data\ u(5,t) = 100$

$When\ x = 5\ and\ t = 0, u = 100\ and\ when\ x = 5, t = 1, u = 100.$

t \ x	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

$Now, by\ Crank - Nicholson\ formula\ we\ calculate\ the\ remaining\ of$

$u_1, u_2, u_3, u_4.$

We use $e = \frac{1}{4}(a + b + c + d)$

$u_1 = \frac{1}{4}(0 + 0 + 20 + u_2) \quad \rightarrow (1)$

$$u_2 = \frac{1}{4}(20 + u_1 + 20 + u_3) \quad \rightarrow (2)$$

$$u_3 = \frac{1}{4}(20 + u_2 + 20 + u_4) \quad \rightarrow (3)$$

$$u_4 = \frac{1}{4}(20 + u_3 + 100 + 100) \quad \rightarrow (4)$$

By simplifying the following equation,

Thus, the final table is

t \ x	0	1	2	3	4	5
0	**0**	**20**	**20**	**20**	**20**	**100**
1	**0**	**9.80**	**2.19**	**30.72**	**59.92**	**100**

$\boldsymbol{b)}$ ***Obtain the Taylor's and Laurent's series which represent the function*** $\boldsymbol{f(z) = \frac{2}{(z-1)(z-2)}}$ ***in the regions***

$\boldsymbol{(1)|z| < 1, (2)\ 1 < |z| < 2.}$ **(6)**

Solution:

$$Let\ \frac{2}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2} \qquad \therefore 2 = a(z-2) + b(z-1)$$

$When\ z = 1, \quad 2 = -a \qquad \therefore \quad a = -2$

$When\ z = 2, \quad 2 = b$

$$\therefore \quad \frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}$$

$Case(1): When\ |z| < 1, clearly\ |z| < 2$

$$\therefore f(z) = \frac{2}{1-z} - \frac{2}{2[1-(1-z/2)]} = 2[1-z]^{-1} - [1-z/2]^{-1}$$

$$\therefore \boldsymbol{f(z) = 2[1 + z + z^2 + z^3 + \cdots] - \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]}$$

$Case(2): When\ 1 < |z| < 2, we\ write$

$$\frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}\ as$$

$$= -\frac{2}{z[1-(1-z)]} - \frac{2}{2[1-(1-z/2)]}$$

$$= -\frac{2}{z}[1-(1/z)]^{-1} - [1-(z/2)]^{-1}$$

$$\boldsymbol{f(z) = -\frac{2}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right] - \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]}$$

$\boldsymbol{c)}$ ***Solve*** $\boldsymbol{(D^2 - 3D + 2)y = 4e^{2t}}$ ***with*** $\boldsymbol{y(0) = -3}$ ***and*** $\boldsymbol{y'(0) = 5}$ ***where***

$\boldsymbol{D = \frac{d}{dt}}.$ **(8)**

Solution:

$Let\ L(y) = \bar{y}. Then\ , taking\ Laplace\ transform,$

$$L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$$

$But\ L(y') = s\bar{y} - y(0) = s\bar{y} + 3$

$and\ L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} + 3s - 5$

$\therefore\ The\ equation\ becomes,$

$$(s^2\bar{y} + 3s - 5) - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} = \frac{4}{s-2} + 14 - 3s = \frac{-3s^2 + 20s - 24}{s-2}$$

$$\therefore\ \bar{y} = \frac{-3s^2 + 20s - 24}{(s-2)(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

$By\ artial\ fraction,\quad \bar{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$Taking\ inverse\ Laplace\ transform,$

$$y = -7L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\left(\frac{1}{s-2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$= -7cL^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s^2}$$

$\boldsymbol{Hence, y = -7e^t + 42e^t + 4te^{2t}}.$

$\boldsymbol{Q.5]\ a)\ Find\ an\ analytical\ function\ f(z) = u + iv, if}$

$\boldsymbol{u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\ \sin y\}}.$ **(6)**

Solution:

$Let\ u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\ \sin y\}$

$$\therefore\ u_x = -e^{-x}\{(x^2-y^2)\cos y + 2xy\ \sin y\} + e^{-x}\{2x\cos y + 2y\ \sin y\}$$

$$= e^{-x}[-(x^2-y^2)\cos y + 2x\cos y + 2y\sin y - 2xysin\ y]$$

$$u_y = e^{-x}[-(x^2-y^2)\sin y - 2y\cos y + 2x\sin y - 2xy\ cos\ y]$$

$$\therefore\ \varphi_1 = u_x, \quad and \quad \varphi_2 = u_y$$

$By\ Milne - Thompson\ method$

$$f'(z) = \varphi_1(z,0) - i\varphi_2\ (z,0) = e^{-z}[-z^2+2z]$$

$$\therefore f(z) = \int e^{-z}[-z^2+2z]dz$$

$Integrating\ by\ parts,$

$$f(z) = (-z^2+2z)(e^{-z}) - \int(-e^{-z})(-2z+2)dz$$

$$= e^{-z}(z^2-2z) + \int e^{-z}(2-2z)dz$$

$Integrating\ by\ parts\ again,$

$$\therefore f(z) = e^{-z}(c-2z) + (2-2z)(-e^{-z}) - \int(-e^{-z})(-2)dz$$

$$= e^{-z}(z^2-2z) - e^{-z}(2-2z) + 2e^{-z}$$

$$= \boldsymbol{e^{-z}e^{-z} + c.}$$

---

b) Find the Laplace transform of

$\boldsymbol{f(t) = t\sqrt{1+\sin t}}$. **(6)**

Solution:

$$\sqrt{1+\sin t} = \sqrt{[sin^2(t/2) + cos^2(t/2) + 2\ sin(t/2)\ cos(t/2)]}$$

$$= \sqrt{[sin(t/2) +\ cos(t/2)]^2} = \ sin(t/2) +\ cos(t/2)$$

$$\therefore\ \sqrt{1+\sin t} = L[sin(t/2) +\ cos(t/2)]$$

$$= \frac{1/2}{s^2+(1/2)^2} + \frac{s}{s^2+(1/2)^2}$$

$$= \frac{1}{2} . \frac{4}{(4s^2+1)} + \frac{4s}{(4s^2+1)}$$

$$= \frac{4s+2}{(4s^2+1)} = \frac{2(2s+1)}{(4s^2+1)}$$

$$\therefore L\left[\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2(2s+1)}{(4s^2+1)}\right] = -2\left[\frac{(4s^2+1)2-(2s+1)8s}{(4s^2+1)^2}\right]$$

$$= -2\frac{[-8s^2-8s+2]}{(4s^2+1)^2}$$

$$= \mathbf{4}\frac{(\mathbf{4s^2+4s-1})}{(\mathbf{4s^2+1})^2}$$

$\boldsymbol{c}$) $\boldsymbol{Obtain\ half\ range\ Fourier\ cosinr\ series\ of\ f(x) = x, 0} < x < 2.$

$\boldsymbol{Using\ Parseval's\ identity\ , deduce\ that} -$

$\boldsymbol{\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4}} + \cdots$ **(8)**

Solution:

$Let\ f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right).$ $\quad Here, l = 2.$

$$\therefore\ a_0 = \frac{1}{l}\int_0^l f(x)dx = \frac{1}{2}\int_0^2 x dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \left[x\frac{\sin(n\pi x/2)}{n\pi/2} + \frac{\cos(n\pi x/2)}{n^2\pi^2/2^2} . 1\right]_0^2$$

$$\therefore\ a_0 = \left[2.(0) + \frac{\cos n\pi}{n^2\pi^2/2^2} - 0 - \frac{1}{n^2\pi^2/2^2}\right]$$

$$= \frac{[(-1)^n - 1]}{n^2\pi^2/2^2}$$

$$= \left\{ -4.\frac{2}{n^2\pi^2} \quad if\ n\ is\ odd \right.$$

$$= \{0 \qquad\qquad if\ n\ is\ even$$

$$\therefore \quad x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right]$$

By Parseval's identity

$$\frac{1}{l}\int_0^l [f(x)]^2 dx = \frac{1}{2}[2a_0^2 + a_1^2 + a_2^2 + \cdots]$$

$$\therefore l.h.s. = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

$$\therefore \quad \frac{4}{3} = \frac{1}{2}\left[2 + \frac{64}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right\}\right]$$

$$\frac{8}{2} - 2 = \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$$\therefore \quad \boldsymbol{\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots}$$

Q.6] a) If $f(a) = \oint_c \frac{3z^2 + 2z + 1}{z - a} dz, C: x^2 + y^2 = 4$ find the values of $f(3), f'(1-i)$ and $f''(1-i)$. **(6)**

Solution:

The circle $x^2 + y^2 = 4$ *is* $|z| = 2$.

1] *The point* $z = 3$ *lies outside the circle* $|z| = 2$.

$\therefore$ *By Cauchy's Integral Theorem*

$\boldsymbol{f(3) = \oint_c \frac{3z^2+2z+1}{z-a} dz = 0}$

2] $The\ point\ z = -1\ i.e.(1, -1)\ lies\ outside\ the\ circle. Hence, we\ take$

$$\varphi(z) = 3z^2 + 2z + 1\ which\ is\ analytical\ everywhere$$

$$\therefore \int_c \frac{\varphi(z)}{z-a}\, dz = 2\pi i \varphi(a) = 2\pi i(3a^2 + 2a + 1)$$

$$\therefore f(\varphi) = \int_c \frac{\boldsymbol{3z^2 + 2z + 1}}{\boldsymbol{z - a}} dz = 2\pi i(3a^2 + 2a + 1)$$

$$\therefore f'(a) = 2\pi i(6z + 2) and\ \ f''(a) = 2\pi i(6)$$

$$\therefore \boldsymbol{f'(1 - i) = 2\pi i[6(1 - i) + 2] = 2\pi i(8 - 6i)}$$

$\boldsymbol{and\ \ f''(1 - i) = 12\pi i.}$

$\boldsymbol{b)}$ ***Find the coefficient of correlation between height of father and height of son from the following data,***

Height of father: **65 66 67 68 69 71 73**

Height of son : **67 68 64 68 72 69 70.** (6)

Solution:

Let x and y denote height of father and height of son respectively.

x	y	$u_i = x_i - 68$	$v_i = y_i - 68$	u_i^2	y_i^2	$u_i v_i$
65	67	-3	-1	9	1	3
66	68	-2	0	4	0	0
67	64	-1	-4	1	16	4
68	68	0	0	0	0	0
69	72	1	4	1	16	4
71	69	3	1	9	1	3
73	70	5	2	25	4	1
	Total	3	2	49	38	24

$Hence, n = 7$

$$Karl\ Pearson's\ coefficient\ of\ correlation$$

$$r_{x,y} = r_{u,v} = \frac{n\sum uv - \sum u \sum v}{\sqrt{n\sum u^2 - (\sum u)^2} - \sqrt{n\sum v^2 - (\sum v)^2}}$$

$$= \frac{7(24) - (3)(2)}{\sqrt{7(49) - (3)^2} - \sqrt{7(38) - (2)^2}}$$

$$= \frac{162}{\sqrt{334}\sqrt{262}}$$

$$= 0.5476$$

$\boldsymbol{Hence, Coefficient\ of\ Correlation\ between\ height\ of\ father\ and\ height\ of\ son\ (r) = 0.5476}$

$\boldsymbol{c)\ A\ tightly\ stretched\ sring\ with\ fixed\ end\ points\ x = 0\ and\ x = l,}$ $\boldsymbol{in\ the\ shape\ defined\ by\ y = kx(l - x)\ where\ k\ is\ a\ constant\ is}$ $\boldsymbol{released\ from\ this\ position\ of\ rest. Find\ y(x,t), the\ vertical}$ $\boldsymbol{displacement\ if\ \frac{\partial^2 y}{\partial t^2} = c\frac{\partial^2 y}{\partial x^2}.}$ **(8)**

Solution:

$The\ vibration\ of\ a\ string\ is\ given\ by\ \frac{\partial^2 y}{\partial t^2} = c\frac{\partial^2 y}{\partial x^2} \quad \rightarrow (1)$

$Since\ the\ vibration\ of\ a\ string\ is\ periodic\ the\ solution\ of\ (1) is\ of\ form$

$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \quad \rightarrow (2)$

$Given\ initial\ boundary\ conditions\ are:$

1) $When\ x = 0, y = 0\ for\ all\ t\ i.e. one\ end\ A\ of\ the\ string\ remains\ fixed\ throughout\ the\ motion.$

$\therefore We\ get\ from\ (2)$

$$0 = (c_1 + 0)(c_3 \cos mct + c_4 \sin mct)$$

$\therefore\ c_1 = 0$

$\therefore\ \ y = c_2 \sin mx\, (c_3 \cos mct + c_4 \sin mct) \qquad \rightarrow (3)$

2) $Now\ \ \partial y/\partial t = 0\ when\ t = 0\ i.e.\ initially\ the\ string\ is\ steady. From\ (3)$

$$\frac{\partial y}{\partial t} = c_2 \sin mx\, \{c_3(-mc) \sin mct + c_4\, (\text{mc}) \cos mct\, \}$$

$Putting\ t = 0, \dfrac{\partial y}{\partial t} = 0, \quad 0 = c_2 \sin mx\, (c_4\, mc) \qquad \therefore c_2 c_4\, mc = 0$

$If\ \ c_2 = 0\ then\ (3) will\ give\ a\ trivial\ solution\ y = 0. \quad \therefore\ \ c_4 = 0$

$Thus, from\ (3), we\ get,$

$y = c_2 c_3 \sin mx \cos mct$

$y = c_5 \sin mx \cos mct \qquad (where\ c_2 c_3 = c_5) \quad \rightarrow (4)$

3) $Now, y = 0\ when\ x = l\ for\ all\ t\ i.e. the\ other\ end\ of\ the\ string\ is$

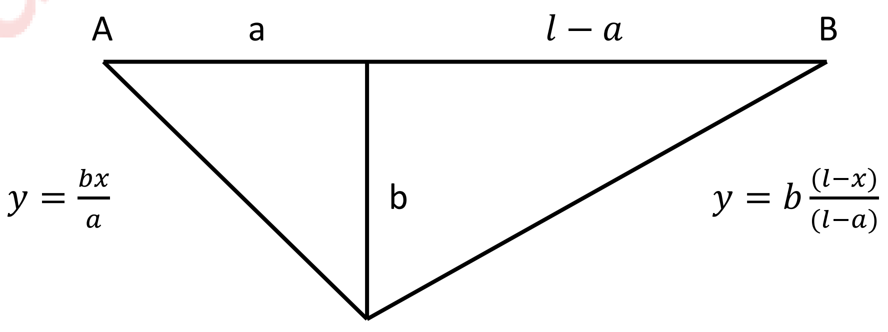
$fixed\ and\ the\ length\ of\ the\ string\ is\ l.$

$\therefore From\ (4), we\ get \quad 0 = c_5 \sin ml \cos mct \qquad \rightarrow (5)$

$If\ \ c_5 = 0\ (4) will\ lead\ to\ a\ trivial\ solution\ y = 0. Thus\ , c_5 cannot\ be\ zero.$

$\therefore From\ (5), we\ get \sin ml = 0$

$\therefore ml = n\pi\ i.e.\ \ m = \dfrac{n\pi}{l}; \qquad n = 1,2,3, \ldots.$



$As\ from\ above, the\ solution\ of\ given\ equation\ is$

$$y = c_5 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \rightarrow (6)$$

$Putting\ n = 1,2,3, \dots$

$$y = \sum b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \rightarrow (7)$$

$Now, lastly\ when\ t = 0, y = kx(l - x)$

$\therefore Putting\ t = 0\ in\ (7) we\ get\ \ y = \sum b_n \sin\frac{n\pi x}{l} \quad \rightarrow (8)$

$where, y\ is\ given\ by\ y = kx(1 - x).$

$But\ the\ series\ (8) is\ a\ Fourier\ half - range\ sine\ series\ for\ the\ function$

$The\ coefficient\ b_n\ can\ be\ determind\ from$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l kx(1 - x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2k}{l}\left[x(1 - x).\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{l}\right) - (1 - 2x)\left(-\frac{l^2}{n^2\pi^2}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-\frac{2l^3}{n^3\pi^3}cosn\pi + \frac{2l^3}{n^3\pi^3}\right] = \frac{4kl^2}{n^3\pi^3}(1 - cosn\pi)$$

$Hence, putting\ the\ value\ of\ b_n\ in\ (8) the\ solution\ is$

$$y = \frac{4kl^2}{\pi^3}\sum\left(\frac{1 - cosnx}{n^3}\right)sin\frac{n\pi x}{l}cos\frac{n\pi ct}{l}, \qquad n = 1,2,3, \dots.$$

$$\therefore\ \boldsymbol{y(x,t) = \frac{8kl^2}{\pi^3}\left[\frac{1}{1^3}sin\frac{\pi x}{l}cos\frac{\pi ct}{l} + \frac{1}{3^3}sin\frac{3\pi x}{l}cos\frac{3\pi ct}{l} + \cdots\right]}$$