MUMBAI UNIVERSITY

SEMESTER 2

APPLIED MATHEMATICS SOLVED PAPER – MAY 2017

N.B:- (1) Question no. 1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1.(a) **Evaluate** $\int_0^\infty 3^{-4x^2}\ dx$ **[3]**

Ans: **Let** I = $\int_0^\infty 3^{-4x^2}\ dx$

put $3^{-4x^2} = e^{-t}$

taking log on both sides,

$$4x^2 \log 3 = t$$

$$x^2 = \frac{t}{4\log 3} \quad => \quad x = \frac{\sqrt{t}}{2\sqrt{\log 3}}$$

diff. w.r.t x,

$$dx = \frac{t^{-1/2}}{4\sqrt{\log 3}} dt \qquad \text{lim}\longrightarrow[\,0, \infty\,]$$

$$\therefore I = \int_0^\infty \frac{e^{-t}}{4\sqrt{\log 3}}\ t^{-1/2}$$

$$\therefore I = \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t}.t^{-1/2}\ dt$$

$$\boxed{\therefore I = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}} \qquad \text{..........}\{\ \int_0^\infty e^{-t}.t^{-1/2}\ dt = \sqrt{\pi}\ \}$$

(b) Solve $(2y^2 - 4x + 5)dx = (y - 2y^2 - 4xy)dy$ **[3]**

Ans : $(2y^2 - 4x + 5)dx = (y - 2y^2 - 4xy)dy$

Compare with Mdx + Ndy = 0

$\therefore$ M = $(2y^2 - 4x + 5)$ $\qquad$ $\therefore$ N = - $(y - 2y^2 - 4xy)$

$\frac{\partial M}{\partial y} = 4y$ $\qquad$ $\frac{\partial N}{\partial x} = 4y$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given diff. eqn is exact .

The solution of exact diff. eqn is given by ,

$$\int M dx + \int [N - \frac{\partial}{\partial y} M dx] \, dy = c$$

$$\int M dx = \int ((2y^2 - 4x + 5))dx = 2xy^2 - 2x^2 + 5x$$

$$\frac{\partial}{\partial y} \int M dx = 4xy$$

$$\int [N - \frac{\partial}{\partial y} M dx] \, dy = \int [4xy - y + 2y^2 - 4xy] dy = \frac{2}{3} y^3 - \frac{y^2}{2}$$

$$\boxed{\therefore 2xy^2 - 2x^2 + 5x + \frac{2}{3} y^3 - \frac{y^2}{2} = c}$$

(c) Solve the ODE $(D-1)^2(D^2+1)^2 y = 0$ [3]

Ans : $(D-1)^2(D^2+1)^2 y = 0$

For complementary solution ,

$$f(D) = 0$$

$$(D-1)^2(D^2+1)^2 = 0$$

$\therefore (D-1)^2 = 0$ $\therefore (D^2+1)^2 = 0$

$D - 1 = 0$ *for two times* $(D^2+1) = 0$ *for two times*

$\therefore$ D -1=0 $\therefore D^2 = -1$

Roots are : D = 1,1,+i,+i,-i,-i

$$\boxed{\therefore y_c = (c_1 + xc_2)e^x + [(c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x]}$$

(d) Evaluate $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$ [3]

Ans : let $I = \int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$

$$= \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right]_0^{x^2} dx$$

$$= \int_0^1 \frac{(e^x - 1)}{\frac{1}{x}} dx$$

$$= \int_0^1 x.e^x\, dx - \int_0^1 x.dx$$

$$= [x.e^x - e^x]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= e - e + 1 - \frac{1}{2}$$

$$\boxed{\therefore I = \frac{1}{2}}$$

(e) Evaluate $\int_0^1 \frac{x^a - 1}{\log x} dx$ [4]

Ans : let $I = \int_0^1 \frac{x^a - 1}{\log x} dx$

Taking 'a' as parameter ,

$$I(a) = \int_0^1 \frac{x^a - 1}{\log x} dx \qquad \text{-------- (1)}$$

differentiate w.r.t a ,

$$\frac{dI(a)}{da} = \frac{d}{da} \int_0^1 \frac{x^a - 1}{\log x} dx$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - 1}{\log x} dx \qquad \text{.........\{ D.U.I.S f(x)\}}$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{x^a .\log x}{\log x} dx \qquad \text{.........}\left\{ \frac{dx^a}{da} = x^a.\log a \right\}$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 x^a\, dx$$

$$\therefore \frac{dI(a)}{da} = \left[\frac{x^{a+1}}{a+1} \right]_0^1$$

$$\therefore \frac{dI(a)}{da} = \frac{1}{a+1} - 0$$

$$\therefore \frac{dI(a)}{da} = \frac{1}{a+1}$$

now , integrate w.r.t a,

$$I(a) = \int \frac{1}{a+1}\, da$$

$$I(a) = \log(a+1) + c \qquad \text{-------- (2)}$$

where c is constant of integration

put a=0 in eqn (1),

$$I(0) = \int_0^1 0\, dx = 0$$

And

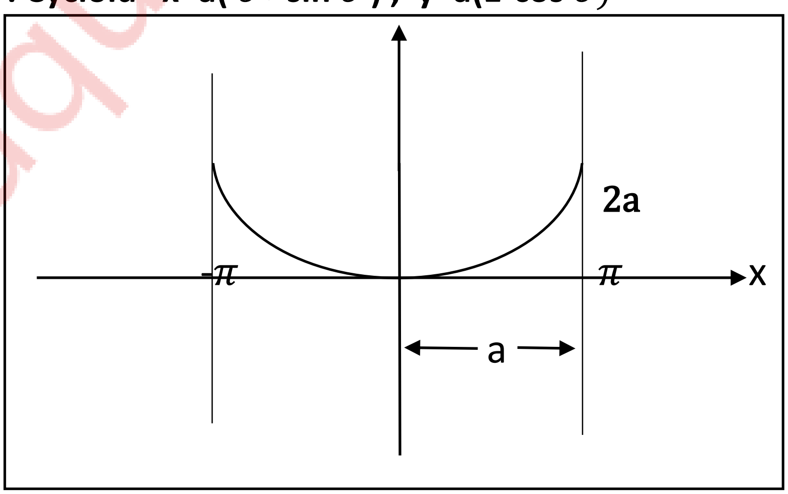
From eqn (2), I(0)= c

$$\therefore c = 0$$

$$\boxed{\therefore I = \log(a+1)}$$

(f) Find the length of cycloid from one cusp to the next , where $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$. **[4]**

Ans : Given curve : Cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$



The length of given curve is :

$$S = \int_{\theta_1}^{\theta_2} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta$$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore (\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = a^2[\,1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\,]$$

$$= 2a^2[\,1 + \cos\theta]$$

$$= 4\,a^2[\,\cos^2 {}^{\theta}\!/_{2}]$$

$$\therefore \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} = 2a\cos {}^{\theta}\!/_{2}$$

$$\therefore S = \int_{-\pi}^{\pi} 2a\cos {}^{\theta}\!/_{2}\, d\theta$$

$$= 2\times \int_{0}^{\pi} 2a\cos {}^{\theta}\!/_{2}\, d\theta$$

$$= 4a\,[\,2\sin {}^{\theta}\!/_{2}\,]_0^{\pi}$$

$$\boxed{\therefore S = 8a}$$

Q.2.(a) Solve $(D^2 - 3D + 2)y = 2e^x \sin(\frac{x}{2})$ **[6]**

Ans : $(D^2 - 3D + 2)y = 2e^x \sin(\frac{x}{2})$

For complementary function ,

$$f(D) = 0$$

$$\therefore (D^2 - 3D + 2) = 0$$

Roots are : D = 2,1 Real roots .

$$\boxed{y_c = c_1 e^x + c_2 e^{2x}}$$

For particular integral ,

$$y_p = \frac{1}{f(D)} X$$

$$= \frac{1}{(D^2-3D+2)} \; 2e^x sin(\frac{x}{2})$$

$$= \; 2e^x \; \frac{1}{(D+1)^2-3(D+1)+2} \; sin(\frac{x}{2})$$

$$= \; 2e^x \; \frac{1}{(D^2-D)} sin(\frac{x}{2})$$

$$= \; 2e^x \frac{1}{-\left(\frac{1}{4}\right)-D} sin(\frac{x}{2})$$

$$= -8e^x \; \frac{1}{4D+1} sin(\frac{x}{2})$$

$$= \; -8e^x \; \frac{4D-1}{16D^2-1} sin(\frac{x}{2})$$

$$\boxed{y_p = \; \frac{8}{5} e^x(-sin\left(\frac{x}{2}\right) - 2cos(\frac{x}{2}))}$$

The general solution of given diff. eqn is given by,

$$\boxed{y_c = y_c + y_p = c_1e^x + c_2e^{2x} + \frac{8}{5} e^x(-sin\left(\frac{x}{2}\right) - 2cos(\frac{x}{2}))}$$

(b) Using D.U.I.S prove that $\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} dx = \frac{\sqrt{\pi}}{2} e^{-2a}, a > 0$ [6]

Ans : Let I(a) = $\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} dx$ ……………….(1)

Taking 'a' as parameter diff. w.r.t. a,

$$\frac{dI(a)}{da} = \frac{d}{da} \int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} dx$$

Apply D.U.I.S rule ,

$$\frac{dI(a)}{da} = \int_0^\infty \frac{\partial}{\partial a} e^{-(x^2+\frac{a^2}{x^2})} dx$$

$$= \int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} . \frac{-2a}{x^2}\, dx$$

Put $\frac{a}{x} = t$, $\frac{-a}{x^2} dx = dt$

Limits [$\infty, 0$**]**

$$\frac{dI(a)}{da} = \int_\infty^0 e^{-(t^2+\frac{a^2}{t^2})} . 2dt = -2\int_0^\infty e^{-\left(t^2+\frac{a^2}{t^2}\right)} dt = -2I(a)$$

$$\frac{dI(a)}{da} = -2I(a)$$

$$\therefore \frac{dI(a)}{I(a)} = -2da$$

Integrating both sides ,

log [I(a)]=-2a +log c

I(a) = c.e^{-2a}

put a=0 in above eqn and eqn (1)

$\therefore$ **I(a) = c =** $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ **..........{ Using gamma function }**

$$\boxed{\therefore I(a) = \frac{\sqrt{\pi}}{2} e^{-2a}}$$

(c) Change the order of integration and evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dx\, dy}{\sqrt{x^2+y^2}}$ **[8]**

Ans : Let I = $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$

Region of integration is : $x \le y \le \sqrt{2-x^2}$

$0 \le x \le 1$

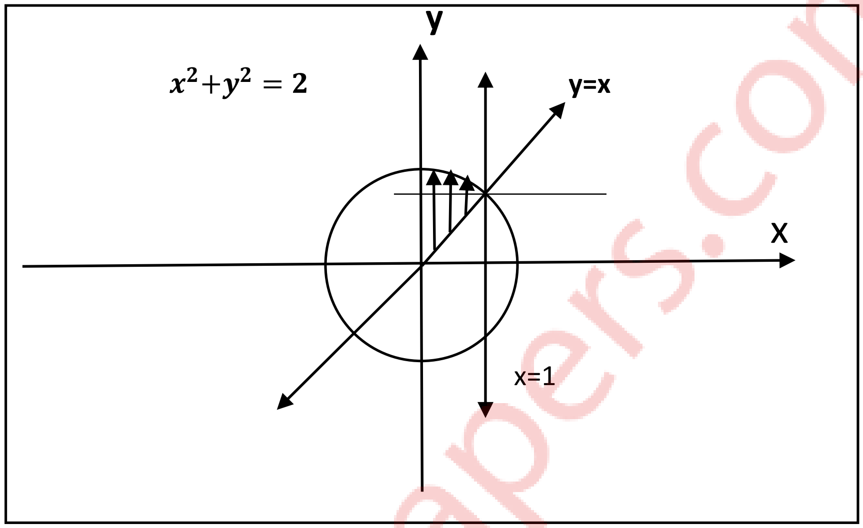
Curves : (i) y = x line

(ii) x=0 , x=1 lines parallel to the y axis .

(iii) y = $\sqrt{2-x^2}$ => $x^2+y^2 = 2$

Circle with centre (0,0) and radius $\sqrt{2}$.

Intersection of circle and y = x line is (1,1) in 1st quadrant.



Divide the region into two parts as shown in fig.

After changing the order of integration :

For one region : $0 \le x \le y$

$0 \le y \le 1$

For another region : $0 \le x \le \sqrt{2-y^2}$

$1 \le y \le \sqrt{2}$

$$\therefore \text{I} = \int_0^1 \int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_0^1 [\sqrt{x^2+y^2}]_0^y \, dy + \int_1^{\sqrt{2}} [\sqrt{x^2+y^2}]_0^{\sqrt{2-y^2}}$$

$$= \int_0^1 (\sqrt{2}.y - y)\,dy + \int_1^{\sqrt{2}} (\sqrt{2}-1)dy$$

$$= (\sqrt{2}-1)[\tfrac{y^2}{2}]_0^1 + [\sqrt{2}y - \tfrac{y^2}{2}]_1^{\sqrt{2}}$$

$= \quad 1 - \frac{1}{\sqrt{2}}$

$$\boxed{\therefore \text{ I } = \frac{\sqrt{2}-1}{\sqrt{2}}}$$

Q.3(a) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dxdydz$ **[6]**

Ans : **Let I** $= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dxdydz$

$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dzdydx$

$= \int_0^1 \int_0^{1-x} \left[\frac{1}{-2(x+y+z+1)^{-2}} \right]_0^{1-x-y}$ dy dx

$= -\int_0^1 \int_0^{1-x} \frac{1}{2} \left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2} \right] dy\, dx$

$= -\int_0^1 \frac{1}{2} \left[\frac{1}{4} y + \frac{1}{(x+y+1)^1} \right]_0^{1-x}$ dx

$= \int_0^1 \frac{1}{2} \left\{ \left[\frac{1}{4}(1-x) - \frac{1}{2}\right] + \left[\frac{1}{x+1}\right] \right\} dx$

$= \frac{1}{2} \left[\frac{1}{4} \left(\frac{(1-x)^2}{8} \right) - \frac{x}{2} + \log(x+1) \right]_0^1$

$$\boxed{\therefore \text{ I } = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]}$$

(b) Find the mass of the lemniscate $r^2 = a^2 \cos 2\theta$ if the density at any point is Proportional to the square of the distance from the pole . [6]

Ans : **Given curve :** $r^2 = a^2 \cos 2\theta$ **is lemniscate.**

The density at any point is proportional to the square of dist. From the pole.

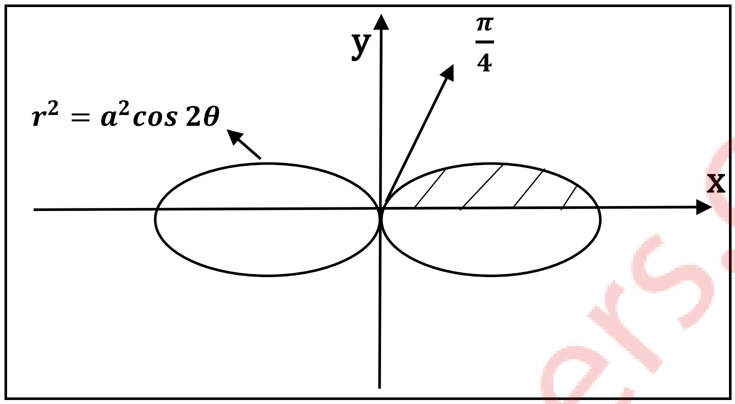
Distance from the pole = r

$\therefore Density \propto r^2$

$\therefore Density = k.r^2$

The mass of the lemniscate is given by ,

$$M = \int_{\theta_1}^{\theta_{21}} \int_{r_1}^{r_2} density\ r\ dr\ d\theta$$



$$\therefore \quad M = 4 \times \int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} k.r^2.r dr d\theta$$

$$= 4k \times \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= k \times \int_0^{\frac{\pi}{4}} a^4.\cos^2 2\theta.d\theta$$

We can solve this definite integral by beta function.

Put $2\theta = t$ => $2\,d\theta = dt$

Limits $[\,0, \frac{\pi}{2}]$

$$\therefore \ M = ka^4 \int_0^{\frac{\pi}{2}} \cos^2 t.\frac{dt}{2}$$

$$= \frac{ka^4}{2} \times \frac{1}{2}\,\beta\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$\boxed{\therefore \ M = \frac{ka^4\pi}{8}}$$

(c) Solve $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4\log x$ **[8]**

Ans : $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{x} = 4\log x$

The given diff. eqn is Cauchy's homogeneous eqn .

Multiply the given eqn by x,

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=4x\log x$$

Put x = e^z log x = z

Diff. w.r.t x,

$\frac{1}{x}=\frac{dz}{dx}$ but $\frac{dy}{dx}=\frac{dy}{dz}.\frac{dz}{dx}=\frac{dy}{dz}\frac{1}{x}$

$$\therefore x\frac{dy}{dx}=Dy$$

$$x^2\frac{d^2y}{dx^2}=D(D-1)y$$

$$x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y \qquad \text{where } D=\frac{d}{dz}$$

$$\therefore \quad [D(D-1)(D-2)+3D(D-1)+D+1]y=4z.e^z$$

$$\therefore \quad [D^3+1]y=4z.e^z$$

For complementary solution ,

$$f(D)=0$$

$$\therefore [D^3+1]=0$$

Roots are: $D=-1,\frac{1}{2}+i\frac{\sqrt{3}}{2},\frac{1}{2}-i\frac{\sqrt{3}}{2}$

Roots of the eqn are real and complex.

$$\boxed{\therefore\ y_c=c_1e^{-z}+e^{z/2}\left(c_2\cos\frac{\sqrt{3}z}{2}+c_3\sin\frac{\sqrt{3}z}{2}\right)}$$

For particular integral ,

$$y_p=\frac{1}{f(D)}X=\frac{1}{(D^3+1)}4z.e^z$$

$$=4e^z\frac{1}{(D+1)^3+1}z$$

$$=4e^z\frac{1}{D^3+3D^2+3D+2}z$$

$$\boxed{\therefore\ y_p = e^z(2z-3)}$$

The general solution of given diff. eqn is ,

$$y_g = y_c + y_p = c_1e^{-z} + e^{z/2}\left(c_2\cos\frac{\sqrt{3}z}{2} + c_3\sin\frac{\sqrt{3}z}{2}\right) + e^z(2z-3)$$

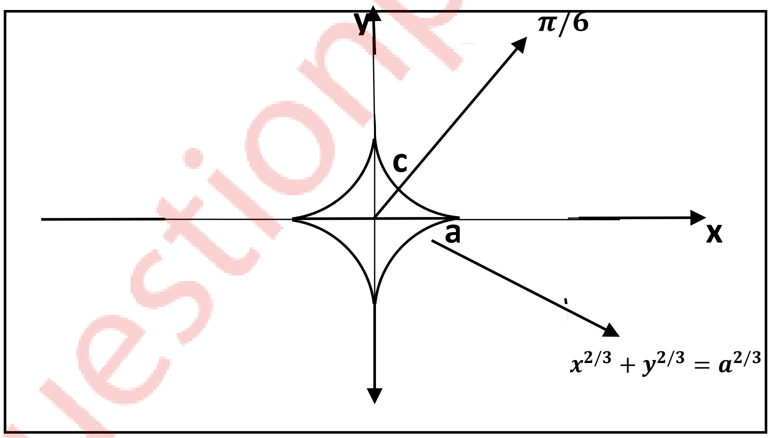
Resubstitute z,

$$\boxed{\therefore y_g = \frac{c_1}{x} + \sqrt{x}\left(c_2\cos\frac{\sqrt{3}\log x}{2} + c_3\sin\frac{\sqrt{3}\log x}{2}\right) + x(2\log x - 3)}$$

Q.4(a) Prove that for an astroid $x^{2/3} + y^{2/3} = a^{2/3}$, the line $\theta = \pi/6$ Divide the arc in the first quadrant in a ratio 1:3. [6]

Ans : Given curve : astroid $x^{2/3} + y^{2/3} = a^{2/3}$

The line $\theta = \pi/6$ cuts the asroid in 1 st quadrant.



C is the point on the curve which cuts the arc.

Length of astroid in first quadrant:

Put x= $a\cos^3 t$ and y=$a\sin^3 t$

dx=-3asin t.$\cos^2 t$dt dy=3acos t.$\sin^2 t$dt

$$S = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^{\pi/2} \sqrt{(-3a\sin t.\cos^2 t)^2 + (3a\cos t.\sin^2 t)^2}\ dt$$

$= \int_0^{\pi/2} 3a.\text{sin t.cost dt}$

$= \frac{3}{2}a \int_0^{\pi/2} \sin 2t\, dt$

$= \frac{3}{4}a\,[-\cos 2t\,]_0^{\pi/2}$

$\boxed{\therefore S = \frac{3}{2}a}$ ....(1)

Now the length of the curve ac : Just put $\frac{\pi}{6}$ *insted of* $\frac{\pi}{2}$ because the curve is

Only upto given line.

$\therefore$ S(ac) = $\int_0^{\pi/6} 3a\, sint\, .\, cost\, dt = \frac{3}{4}a\,[-\cos 2t\,]_0^{\pi/6}$

$= \frac{3}{4}a\,[-\frac{1}{2} + 1\,]$

$\boxed{S(ac) = \frac{3}{8}a}$ ....(2)

Legnth of remaining part $= \frac{3}{2}a - \frac{3}{8}a = \frac{9}{8}a$ ....(3)

Divide eqn (3) and (2).

The line $\frac{\pi}{6}$ cuts the given astroid in the ratio of 1:3

Hence proved.

(b) Solve $(D^2 - 7D - 6)y = (1 + x^2)e^{2x}$ **[6]**

Ans : $(D^2 - 7D - 6)y = (1 + x^2)e^{2x}$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 - 7D - 6) = 0$$

Roots are : $D = \frac{7}{2} + \frac{\sqrt{73}}{2}\ ,\ \frac{7}{2} - \frac{\sqrt{73}}{2}$

Roots of the given diff. eqn are irrational roots .

$$\boxed{y_c = e^{\frac{7x}{2}}(c_1 cosh\frac{\sqrt{73}}{2} + c_2 sinh\frac{\sqrt{73}}{2})}$$

For particular integral,

$$y_p = \frac{1}{f(D)}\, X$$

$$= \frac{1}{(D^2-7D-6)}[e^{2x} + e^{2x}x^2]$$

$$= \frac{1}{(D^2-7D-6)}e^{2x} + \frac{1}{(D^2-7D-6)}e^{2x}x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}\frac{1}{(D+2)^2-7(D+2)-6}x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}\frac{1}{D^2-3D-16}x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}[\frac{1}{-16}(\frac{1}{1+\frac{3D-D^2}{16}})\ ]x^2$$

$$= -\frac{e^{2x}}{16} + e^{2x}[\frac{1}{-16}(\frac{1}{1+\frac{3D-D^2}{16}})\ ]x^2$$

$$= -\frac{e^{2x}}{16}[\,1 + (1+\frac{3D-D^2}{16})^{-1}x^2\,]$$

$$= -\frac{e^{2x}}{16}\{\ 1 + [1 - \frac{3D-D^2}{16} + (\frac{3D-D^2}{16})^2\ ]\,x^2\ \}$$

$$= -\frac{e^{2x}}{16}\{\,1 + [x^2 - \frac{3}{8}x + \frac{2}{16} + \frac{9}{16\times 8}\ ]\,\}$$

$$= -\frac{e^{2x}}{16}\{\,1 + [x^2 - \frac{3}{8}x + \frac{25}{128}\ ]\,\}$$

$$\boxed{y_p = -\frac{e^{2x}}{16} - \frac{e^{2x}}{16}[x^2 - \frac{3}{8}x + \frac{25}{128}\ ]}$$

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c + y_p = e^{\frac{7x}{2}}(c_1 cosh\frac{\sqrt{73}}{2} + c_2 sinh\frac{\sqrt{73}}{2}) - \frac{e^{2x}}{16} - \frac{e^{2x}}{16}[x^2 - \frac{3}{8}x + \frac{25}{128}]}$$

(c) Apply Rungee Kutta method of fourth order to find an approximate Value of y when x=0.4 given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y = 1$ *when* $x = 0$ Taking h=0.2. **[8]**

Ans : (I) $\frac{dy}{dx} = \frac{y-x}{y+x}$ $\quad x_0 = 0, y_0 = 1, h = 0.2$

$$f(x,y) = \frac{y-x}{y+x}$$

$$k_1 = h.f(x_0, y_0) = 0.2f(0,1) = 0.2$$

$$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2.f(0.1, 1.1) = 0.1666$$

$$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2.f(0.1, 1.0833) = 0.1661$$

$$k_4 = h.f(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1661) = 0.1414$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{0.2 + 2(0.1666) + 2(0.1661) + 0.1414}{6} = 0.1678$$

$$\boxed{\therefore \ y(0.2) = y_0 + k = 1 + 0.1678 = 1.1678}$$

(II) $x_1 = 0.2, y_2 = 1.1678, h = 0.2$

$$k_5 = h.f(x_1, y_1) = 0.2f(0.2, 1.1678) = 0.1415$$

$$k_6 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_5}{2}\right) = 0.2.f(0.3, 1.23855) = 0.1220$$

$$k_7 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_6}{2}\right) = 0.2.f(0.3, 1.2285) = 0.1214$$

$$k_8 = h.f(x_1 + h, y_1 + k_7) = 0.2f(0.4, 1.2892) = 0.1052$$

$$k* = \frac{k_5 + 2k_6 + 2k_7 + k_8}{6} = \frac{0.1415 + 2(0.1220) + 2(0.1215) + 0.1052}{6} = 0.1222$$

$$\boxed{y(0.4) = y_1 + k* = 1.1678 + 0.1222 = 1.290}$$

Q.5(a) Use Taylor series method to find a solution of $\frac{dy}{dx} = xy + 1 , y(0) = 0$

X=0.2 taking h=0.1 correct upto 4 decimal places. [6]

Ans : (I) $\frac{dy}{dx} = xy + 1$,$x_0 = 0, y_0 = 0,$ **h=0.1**

$$f(x,y) = 1 + xy$$

$y' = 1 + xy$ $\quad$ $y'_0 = 1$

$y'' = xy' + y$ $\quad$ $y''_0 = 0$

$y''' = xy'' + 2y'$ $\quad$ $y'''_0 = 2$

Taylor's series is given by ,

y(0.1) = $y_0 + h.y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots$

= 0 + 0.1(1) + 0 + $\frac{(0.1)^3}{6}(2)$

$\boxed{\textbf{y(0.1) = 0.1003}}$

(II) $x_1 = 0.1, y_1 = 0.1003$,**h=0.1**

$y' = 1 + xy$ $\quad$ $y'_0 = 1.01003$

$y'' = xy' + y$ $\quad$ $y''_0 = 0.201303$

$y''' = xy'' + 2y'$ $\quad$ $y'''_0 = 2.0401903$

∴ y(0.2) = 0.1003+1.01003(0.1)+$\frac{0.1^2}{2!}(0.201303) + \frac{0.1^3}{6}(2.0401903)$

$\boxed{\textbf{∴ y(0.2) = 0.202708}}$

(b) Solve by variation of parameters $\left(\frac{d^2y}{dx^2} + 1\right)y = \frac{1}{1+\sin x}$ **[6]**

Ans : put $\frac{d}{dx} = D$

$$(D^2 + 1)y = \frac{1}{1+\sin x}$$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 + 1) = 0$$

Roots are : D $= i , -i$

Roots of given diff. eqn are complex.

The complementary solution of given diff. eqn is given by,

$$\boxed{\therefore\ y_c = c_1 \cos x + c_2 \sin x}$$

For particular solution ,

By method of variation of parameters,

$$y_p = y_1 p_1 + y_2 p_2$$

where $p_1 = \int \frac{-y_2 X}{w}\, dx$

$$p_2 = \int \frac{y_1 X}{w}\, dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$p_1 = \int \frac{-y_2 X}{w}\, dx = \int -\frac{\sin x}{1} . \frac{1}{1+\sin x}\, dx = -\int \frac{\sin x}{1+\sin x} \frac{(1-\sin x)}{(1-\sin x)} dx$$

$$= -\int (\sec x . \tan x - \tan^2 x) dx$$

$$= -[\sec x - \tan x + x]$$

$$p_2 = \int \frac{y_1 . X}{w}\, dx = \int \frac{\cos x}{1} \frac{1}{1+\sin x}\, dx = \log(1 + \sin x)$$

$$y_p = -[\sec x - \tan x + x] \cos x + \log(1 + \sin x) \sin x$$

The general solution of given diff. eqn is given by ,

$$\boxed{y_g = y_c + y_p = c_1 cosx + c_2 sinx - [\sec x - \tan x + x]\cos x + log\,(1 + sinx)sinx}$$

(c) Compute the value of $\int_{0.2}^{1.4}(sin\ x - ln\ x + e^x)dx$ using (i) Trapezoidal Rule (ii) Simpson's (1/3)rd rule (iii) Simpson's (3/8)th rule by dividing Into six subintervals. [8]

Ans : let I = $\int_{0.2}^{1.4}(sin\ x - ln\ x + e^x)dx$

Dividing limits in six subintervals .

$\therefore$ n = 6 $\therefore$ h = $\frac{b-a}{n} = \frac{1.4-0.2}{6} = \frac{1}{5}$

$x_0 = 0.2$	$x_1 = 0.4$	$x_2 = 0.6$	$x_3 = 0.8$	$x_4 = 1.0$	$x_5 = 1.2$	$x_6 = 1.4$
$y_0 = 3.02$	$y_1 = 2.79$	$y_2 = 2.89$	$y_3 = 3.16$	$y_4 = 3.55$	$y_5 = 4.06$	$y_6 = 4.4$

(i) Trapezoidal rule : I = $\frac{h}{2}[\,X + 2R]$ -----------------(1)

$$X = sum\ of\ extreme\ ordinates = 7.42$$

$$R = sum\ of\ remaining\ ordinates = 16.45$$

I = $\frac{1}{5\times 2}(7.42 + 2(16.45))$ ....(from 1)

$\boxed{I = 4.032}$

(ii) Simpson's $(1/3)^{rd}$ rule :

I = $\frac{h}{3}[\,X + 2E + 4O\,]$ ---------------(2)

$$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 4.4 + 3.02 = 7.42$$

$$E = sum\ of\ even\ base\ ordinates = y_2 + y_4 = 6.44$$

$$O = sum\ of\ odd\ base\ ordinates = y_1 + y_3 + y_5 = 10.01$$

I = $\frac{1}{3\times 5}(7.42 + 2\times 6.44 + 4\times 10.01)$ ....(from 2)

$$\boxed{I = 4.022}$$

(iii) Simpson's $(3/8)^{th}$ rule :

$$I = \frac{3h}{8}[\,X + 2T + 3R\,] \qquad \text{-------------(3)}$$

$$X = sum\ of\ extreme\ ordinates = y_0 + y_6 = 4.4 + 3.02 = 7.42$$

$$T = sum\ of\ multiple\ of\ three\ base\ ordinates = y_3 = 3.16$$

$$R = sum\ of\ remaining\ ordinates = y_1 + y_2 + y_4 + y_5 = 13.49$$

$$\therefore I = \frac{3\times1}{8\times5}[\,7.42 + 2\times3.16 + 3\times13.49\,]$$

$$\boxed{\therefore I = 4.02075}$$

Q.6(a). Using beta functions evaluate $\int_0^{\pi/6} cos^6 3\theta . sin^2 6\theta d\theta$ [6]

Ans : let $I = \int_0^{\pi/6} cos^6 3\theta . sin^2 6\theta d\theta$

Put $3\theta = t$

Diff. w.r.t θ,

$d\theta = \frac{dt}{3}$ limits : $[0, \frac{\pi}{2}]$

$$\therefore I = \frac{1}{3}\int_0^{\pi/2} cos^6 t . sin^2 2t dt$$

$$= \frac{4}{3}\int_0^{\pi/2} cos^3 t\,(sin\,t . cos\,t)^2 dt$$

$$= \frac{4}{3}\int_0^{\pi/2} cos^5 t . sin^2 t . dt$$

$$= \frac{4}{3} \times \frac{1}{2} \times \beta(3, \tfrac{3}{2}) \qquad ..\{\int_0^{\pi/2} cos^m t . sin^n t . dt = \tfrac{1}{2} \times \beta(m+1, n+1)\}$$

$$\boxed{\therefore I = \frac{32}{315}}$$

(b) Evaluate $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} log(x^2+y^2)dxdy$ **by changing to polar Coordinates.** **[6]**

Ans : **let** **I =** $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} log(x^2+y^2)dxdy$

Region of integration : $y \le x \le \sqrt{a^2-y^2}$

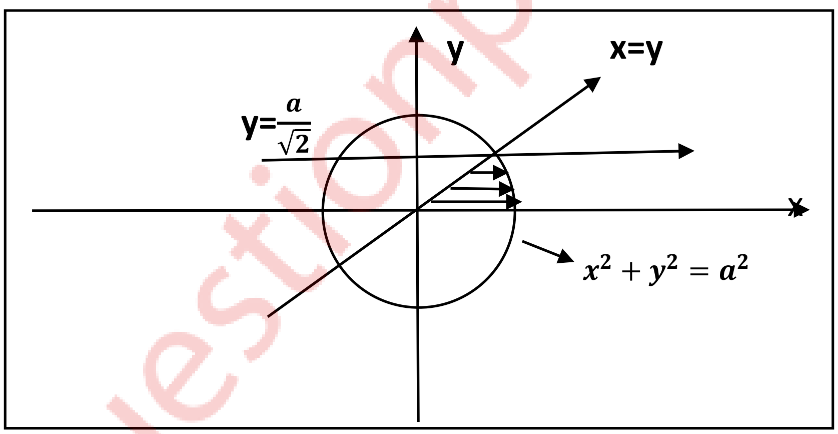
$$0 \le y \le \frac{a}{\sqrt{2}}$$

The line x=y is inclined at 45° to the +ve x-axis.

Curves : (i) x=y , y=0 , y=$\frac{a}{\sqrt{2}}$ **lines**

(ii) $x = \sqrt{a^2-y^2}$

$x^2+y^2=a^2$ **circle with centre (0,0) and radius a.**



Cartesian coordinates ⟶ **Polar coordinates**

(x,y) ⟶ **(r,θ)**

Put x = r cos θ and y = r sin θ

$$f(x,y) = log(x^2+y^2) = log\, r^2 = 2log\, r = f(r,\theta)$$

Limits changes to : $0 \le r \le a$

$$0 \le \theta \le \frac{\pi}{4}$$

$\therefore$ I = $\int_0^{\frac{\pi}{4}} \int_0^{a} 2 log\, r . r\, dr\, d\theta$

= $2\int_0^{\frac{\pi}{4}} [\, log\, r . \frac{r^2}{2} - \frac{r^2}{4}]\,{}_0^{a}\, d\theta$

= $2\int_0^{\frac{\pi}{4}} \left[\, log\, a . \frac{a^2}{2} - \frac{a^2}{4}\right] d\theta$

$$\boxed{\therefore I = \left[\, log\, a . \frac{a^2}{2} - \frac{a^2}{4}\right] \times \frac{\pi}{4}}$$

(c) Evaluate $\int\int\int x^2 yzdxdydz$ over the volume bounded by planes x=0,y=0, z=0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ [8]

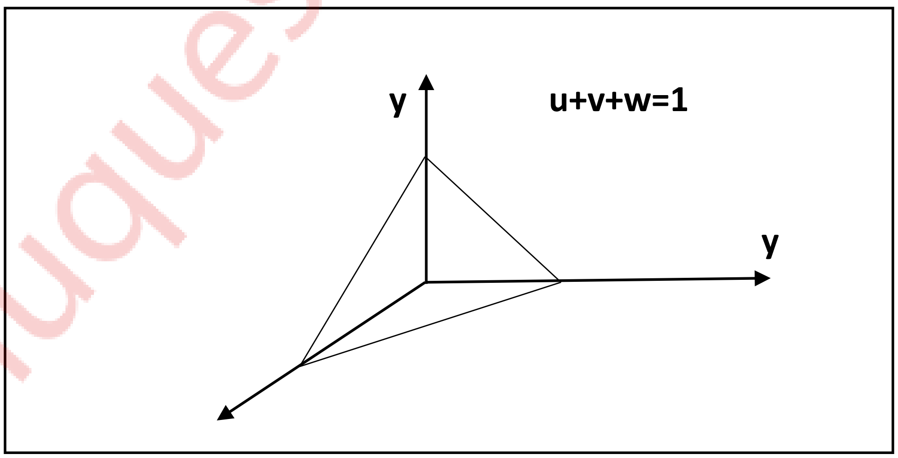
Ans : Let V = $\int\int\int x^2 dxdydz$

Region of integration is volume bounded by the planes x=0,y=0,z=0

And $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Put x = au , y = bv , z = cw

$\therefore$ dxdydz = abc du.dv



The intersection of tetrahedron with all axes is : (1,0,0),(0,1,0),(0,0,1).

$$0 \le w \le (1-u-v)$$

$$0 \le v \le (1-u)$$

$$0 \le u \le 1$$

The volume required is given by ,

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc\, a^2u^2bv.cw.dudvdw$$

$$= \frac{1}{2}a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2v\,(1-u-v)^2 dvdu$$

$$= \frac{1}{2}a^3b^2c^2 \int_0^1 \int_0^{1-u} u^2v[(1-u)^2 - 2(1-u)v + v^2]dudv$$

$$= \frac{1}{2}a^3b^2c^2 \int_0^1 u^2[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}]\begin{matrix}1-u\\0\end{matrix}\ du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \frac{u^2(1-u)^4du}{12}$$

$$= \frac{a^3b^2c^2}{24}\,\beta(3,5)$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{2!4!}{7!}\right)$$

$$\boxed{\therefore I = \frac{a^3b^2c^2}{2520}}$$