MUMBAI UNIVERSITY CBCGS SEM I

Q.P. Code : 77691

APPLIED MATHS I DEC 2019 PAPER SOLUTIONS

Q1)a) If $\sin\left(\theta+i\varphi\right)=\tan\alpha+i\sec\alpha$ **, then show that** $\cos 2\theta.\cosh 2\varphi=3$ **.** **(5M)**

Ans : We have $\sin\left(\theta+i\varphi\right)=\tan\alpha+i\sec\alpha$

$$\therefore \sin\alpha\cos i\varphi+\cos\theta\sin i\varphi=\tan\alpha+i\sec\alpha$$

$$\therefore \sin\alpha\cosh\varphi+i\cos\theta\sinh\varphi=\tan\alpha+i\sec\alpha$$

Equating real and imaginary parts ,

$$\begin{aligned}\tan\alpha &= \sin\theta\cosh\varphi \\ \sec\alpha &= \cos\theta\sinh\varphi\end{aligned}$$

But

$$\sec^2\alpha-\tan^2\alpha=1$$
$$\therefore \cos^2\theta\sinh^2\varphi-\sin^2\theta\cosh\varphi=1$$
$$\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\varphi-1}{2}\right)-\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cosh 2\varphi}{2}\right)=1$$
$$\therefore \cosh 2\varphi-1+\cos 2\theta\cosh 2\varphi-\cos 2\theta-1-\cosh 2\varphi+\cos 2\theta+\cos 2\theta\cosh 2\varphi=4$$
$$\therefore 2\cos 2\theta\cosh 2\varphi=6$$
$$\therefore \cos 2\theta\cosh 2\varphi=3$$

Q1)b) If $u=\log(\tan x+\tan y)$ **, then show that** $\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2$ **.** **(5M)**

Ans : We have

$$\frac{\partial u}{\partial x}=\frac{1}{(\tan x+\tan y)}\sec^2 x$$

$$\therefore \sin 2x\frac{\partial u}{\partial x}=2\sin x\cos x\frac{1}{(\tan x+\tan y)}\sec^2 x=2.\frac{\tan x}{\tan x+\tan y}$$

Similarly , $\sin 2y\frac{\partial u}{\partial y}=2.\frac{\tan y}{(\tan x+\tan y)}$

$$\therefore \sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2\frac{\tan x+\tan y}{(\tan x+\tan y)}=2\,.$$

Similarly , prove that

$$\therefore \sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$$

Q1)c) Express the matrix $A=\begin{bmatrix}0&5&-3\\1&1&1\\4&5&9\end{bmatrix}$ **as the sum of a symmetric and skew symmetric matrix.**

(5M)

Ans : We have

$$A'=\begin{bmatrix}0&1&4\\5&1&5\\-3&1&9\end{bmatrix}$$

$$\therefore A+A'=\begin{bmatrix}0&6&1\\6&2&6\\1&6&18\end{bmatrix}$$

$$A-A'=\begin{bmatrix}0&4&-7\\-4&0&-4\\7&4&0\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A')$ and $Q=\frac{1}{2}(A-A')$.

But we know that P is symmetric and Q is skew-symmetric and A=P + Q .

$$\therefore A=P+Q=\begin{bmatrix}0&3&1/2\\3&1&3\\1/2&3&9\end{bmatrix}+\begin{bmatrix}0&2&-7/2\\-2&0&-2\\7/2&2&0\end{bmatrix}$$

The first matrix is symmetric and the second is skew-symmetric .

Q1)d) Expand $\sqrt{1+\sin x}$ in ascending powers of x upto x^4 terms . **(5M)**

Ans : We have

$$\sqrt{1+\sin x}=\sqrt{\sin^2(x/2)+\cos^2(x/2)+2\sin(x/2)\cos(x/2)}$$

$$=\sqrt{\left[\sin(x/2)+\cos(x/2)\right]^2}=\sin(x/2)+\cos(x/2)$$

$$=\left(\frac{x}{2}\right)-\frac{1}{6}\left(\frac{x}{2}\right)^3+.........+1-\frac{1}{2}\left(\frac{x}{2}\right)^2+\frac{1}{24}\left(\frac{x}{4}\right)^4-....$$

$$=\frac{x}{2}-\frac{x^3}{48}+.......+1-\frac{x^2}{8}+\frac{x^4}{384}-.......$$

$$=1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}+\frac{x^4}{384}-.......$$

Q2)a) Find non-singular matrices P and Q such that PAQ is in normal form where,

$A=\begin{bmatrix}4&3&1&6\\2&4&2&2\\12&14&5&16\end{bmatrix}$**. Also find the rank of A .** **(6M)**

Ans : We first write

$$A=I_3AI_4$$

$$\begin{bmatrix}4&3&1&6\\2&4&2&2\\12&14&5&16\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_1\to\frac{R_1}{4},$

$$\begin{bmatrix}1&3&1&6\\0.5&4&2&2\\3&14&5&16\end{bmatrix}=\begin{bmatrix}0.5&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_2\to R_2-2R_1,R_3-3R_1$

$$\begin{bmatrix}1&3&1&6\\0&-3&-2&-5\\0&-6&-4&-22\end{bmatrix}=\begin{bmatrix}0.5&0&0\\-2&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_2 \to C_2 - 2C_1, C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\frac{-C_2}{3}, \frac{-C_3}{2}, -\frac{C_5}{5}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 3/2 & 4/5 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/5 \end{bmatrix}.$$

$C_3 - C_2, C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 5/6 & -7/24 \\ 0 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1/2 & -5/24 \\ 0 & 0 & 0 & -1/12 \end{bmatrix}.$$

C_{34}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -7/24 & 5/6 \\ 0 & -1/3 & 0 & 1/3 \\ 0 & 0 & -5/24 & 1/2 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}.$$

Q2)b) If $z = f(x,y)$ **and** $x = u\cosh v, y = u\sinh v$,prove that

$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2$. **(6M)**

Ans : We have ,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cosh v + \frac{\partial z}{\partial y}\sinh v \qquad \text{-------- (1)}$$

And , $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}.\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}.\dfrac{\partial y}{\partial v} = \dfrac{\partial z}{\partial x}\text{usinh}\, v + \dfrac{\partial z}{\partial y}\text{ucosh}\, v$

$\dfrac{1}{u}.\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\sinh v + \dfrac{\partial z}{\partial y}\cosh v$ --------- (2) .

Now squaring (1) and (2) and subtracting , we get

$$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}.\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cosh^2 v + \left(\frac{\partial z}{\partial y}\right)^2\sinh^2 v + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\text{coshv.sinh}\, v - \left(\frac{\partial z}{\partial x}\right)^2\sinh^2 v - \left(\frac{\partial z}{\partial y}\right)^2\cosh^2 v - 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\text{coshv.sinh}\, v$$

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 .$$

Q2)c) Prove that $\log\left(\dfrac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right) = i\left(2n\pi + \tan^{-1}\dfrac{2ab}{a^2-b^2}\right)$ **.Hence evaluate** $\log\left(\dfrac{1+5i}{5+i}\right)$**.**

(6M)

Ans : Let a + b=A , a – b =B .

$$\therefore \log\left[\frac{B+iA}{A+iB}\right] = 2n\pi i + \log\left[\frac{B+iA}{A+iB}\right]$$
$$= 2n\pi i + \log(B+iA) - \log(A+iB)$$
$$= 2n\pi i + \left[\log\sqrt{B^2+A^2} + i\tan^{-1}\frac{A}{B}\right] - \left[\log\sqrt{A^2+B^2} + i\tan^{-1}\frac{B}{A}\right] .$$
$$= 2n\pi i + i\left[\tan^{-1}\frac{A}{B} - \tan^{-1}\frac{B}{A}\right]$$

But $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

$$\therefore \log\left[\frac{B+iA}{A+iB}\right] = 2n\pi i + i\tan^{-1}\left[\frac{(A/B)-(B/A)}{1+(A/B)*(B/A)}\right] .$$
$$= 2n\pi i + i\tan^{-1}\left(\frac{A^2-B^2}{2AB}\right)$$

But $A^2 - B^2 = (a+b)^2 - (a-b)^2 = 4ab$ and $AB = (a+b)(a-b) = a^2 - b^2$.

$$\therefore \log\left[\frac{B+iA}{A+iB}\right] = 2n\pi i + i\tan^{-1}\frac{4ab}{2(a^2-b^2)} = i\left[2n\pi + \tan^{-1}\frac{2ab}{(a^2-b^2)}\right]$$

Q3)a) If α and β are the roots of the equation $z^2\sin^2\theta - z\sin 2\theta + 1 = 0$**, then prove that** $\alpha^n + \beta^n = 2\cos n\theta \cos ec^n\theta$ and $\alpha^n\beta^n = \cos ec^{2n}\theta$. **(6M)**

Ans : Solving the quadratic equation in z ,

$$z = \frac{\sin 2\theta \pm \sqrt{\sin^2 2\theta - 4\sin^2\theta}}{2\sin^2\theta} = \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$\therefore z = \frac{\cos\theta \pm \sqrt{\cos^2\theta - 1}}{\sin\theta} = \frac{\cos\theta \pm \sqrt{-\sin^2\theta}}{\sin\theta}$$

$$= \frac{\cos\theta \pm i\sin\theta}{\sin\theta} = (\cos\theta \pm i\sin\theta)\,\text{cosec}\,\theta$$

Let $\alpha = (\cos\theta + i\sin\theta)\,\text{cosec}\,\theta$, $\beta = (\cos\theta - i\sin\theta)\,\text{cosec}\,\theta$.

$$\therefore \alpha^n = (\cos\theta + i\sin\theta)^n\,\text{cosec}^n\,\theta = (\cos n\theta + i\sin(n\theta))\,\text{cosec}^n\,\theta$$

$$\beta^n = (\cos\theta - i\sin\theta)^n\,\text{cosec}^n\,\theta = (\cos n\theta - i\sin(n\theta))\,\text{cosec}^n\,\theta$$

$$\therefore \alpha^n + \beta^n = 2\cos n\theta\,\text{cosec}^n\,\theta$$

$$\alpha^n.\beta^n = (\cos\theta + i\sin\theta)^n\,\text{cosec}^n\,\theta.(\cos\theta - i\sin\theta)^n\,\text{cosec}^n\,\theta$$

$$= (\cos^2 n\theta + \sin^2 n\theta)\,\text{cosec}^{2n}\,\theta = \cos ec^{2n}\theta$$

Q3)b) Solve the following equations by Gauss-Siedal Method ;

15x+2y+z = 18 , 2x+20y-3z = 19 , 3x-6y+25z = 22 .Take three iterations . **(6M)**

Ans : We first write the equations as

$$x = \frac{1}{15}[18 - 2y - z] \quad \text{-------------------(1)}$$

$$y = \frac{1}{20}[19 - 2x + 3z] \quad \text{------------------(2)}$$

$$z = \frac{1}{25}[22 - 3x + 6y] \quad \text{-------------------(3)}$$

(i) First Iteration :We start with the approximation $y_0 = 0, z_0 = 0$ and then from (1) , we get

$$x_1 = \frac{18}{15} = 1.2 \quad .$$

We use this approximation to find y from (2), i.e we put x=1.2 and z=0 in (2) and get

$$y_1 = \frac{1}{20}\left[19 - 2(1.2) + 3(0)\right] = \frac{16.6}{20} = 0.83 \quad .$$

We use these values of x and y to find z from (3) i.e. we put x=1.2 , y=0.83 in (3) and get

$$z = \frac{1}{25}\left[22 - 3(1.2) + 6(0.83)\right] = \frac{23.38}{25} = 0.9352 \quad .$$

(ii) Second iteration : We use the latest values of y and z in (1) to find x , i.e we put y=0.83 and z=0.9352 in (1) and get

$$x_2 = \frac{1}{15}\left[18 - 2(0.83) - 0.9352\right] = \frac{15.4048}{15} = 1.0270 \quad .$$

We now put x=1.027 and z=0.9352 in (2) and get

$$y_2 = \frac{1}{20}\left[19 - 2(1.027) + 3(0.9352)\right] = \frac{19.7516}{20} = 0.9876 \quad .$$

We use these values of x and y to find z from (3)

$$z_2 = \frac{1}{25}\left[22 - 3(1.0270) + 6(0.9876)\right] = \frac{28.8446}{25} = 0.9938 \quad .$$

(iii) Third Iteration : We use the latest values of y and z to find x , i.e. ,we put 0.9876 and z=0.9938 in (1) and get

$$x_3 = \frac{1}{15}\left[18 - 2(0.9876) - 0.9938\right] = \frac{15.031}{15} = 1.0021 \quad .$$

We now put x=1.0021 and z=0.9938 in (2) and get

$$y_3 = \frac{1}{20}\left[19 - 2(1.0021) + 3(0.9938)\right] = \frac{19.9772}{20} = 0.9989 \quad .$$

We use these values of x and y to find z from (3)

$$z_3 = \frac{1}{25}\left[22 - 3(1.0021) + 6(0.9989)\right] = \frac{24.9871}{25} = 0.9995 \quad .$$

Hence , we get x =1.0021 , y =0.9989 , z =0.9995 .

Q3)c) Prove that if z is a homogeneous function of two variables x and y of degree n , then $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$ **.Hence find the value of** $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}$ **at x=1 , y=1 when** $z=x^6\tan^{-1}\left(\frac{x^2+y^2}{x^2+xy}\right)+\frac{x^4+y^4}{x^2+y^2}$ **.** **(8M)**

Ans : Since z is a homogeneous function of degree n in x and y , by Euler's Theorem ,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \qquad \text{-------------- (1)}$$

Differentiating (1) partially w.r.t x ,

$$\left(x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}.1\right)+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$

Differentiating (1) partially w.r.t y ,

$$x\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$

Multiplying (2) by x and (3) by y and adding , we get ,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=(n-1)nz$$

Further, if u is a homogeneous function of three variables x , y , z of degree n then we can prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xy\frac{\partial^2 u}{\partial y\partial z}+2xy\frac{\partial^2 u}{\partial z\partial x}=n(n-1)\,\mathrm{u}$$

For $z=x^6\tan^{-1}\left(\frac{x^2+y^2}{x^2+xy}\right)+\frac{x^4+y^4}{x^2+y^2}$

Putting X=xt , Y=yt , we get

F(X,Y) = $X^6\tan^{-1}\left(\frac{X^2+Y^2}{X^2+XY}\right)+\frac{X^4+Y^4}{X^2+Y^2}$

$$\therefore f(X,Y) = x^6 t^6 \tan^{-1}\left(\frac{x^2t^2 + y^2t^2}{x^2t^2 + xtyt}\right) + \frac{x^4t^4 + y^4t^4}{x^2t^2 + y^2t^2} = x^6 t^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + xy}\right) + t^2 \frac{x^4 + y^4}{x^2 + y^2}$$.

Now, let $x^6 t^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + xy}\right)$ = u , and $t^2 \frac{x^4 + y^4}{x^2 + y^2}$ = v .

u and v are homogeneous functions of degree 6 and 2 respectively .

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u \ , \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)nu = 30u$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = (n-1)nv = 2v$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = (n-1)nu + n(n-1)v = 30u + 2v$$.

$$\therefore x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)nz = 30x^6 t^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + xy}\right) + 2t^2 \frac{x^4 + y^4}{x^2 + y^2}$$.

Q4)a) If $\tan(\alpha + i\beta) = \cos\theta + i\sin\theta$ **then prove that** $\alpha = \frac{n\pi}{2} + \frac{\pi}{4}, \beta = \frac{1}{2}\log\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$**.** **(6M)**

Ans : We have , $\tan(\alpha + i\beta) = \cos\theta + i\sin\theta \quad \therefore \tan(\alpha - i\beta) = \cos\theta - i\sin\theta$

$$\therefore \tan 2\alpha = [\tan((\alpha + i\beta) + (\alpha - i\beta))]$$

$$= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta)\tan(\alpha - i\beta)} = \frac{2\cos\theta}{1 - (\cos^2\theta + \sin^2\theta)}$$

$$\tan 2\alpha = \frac{2\cos\theta}{0}$$

$$\because 2\alpha = n\pi + \frac{\pi}{2}, \alpha = \frac{n\pi}{2} + \frac{\pi}{4}$$

Also , $\therefore \tan 2\beta = [\tan((\alpha + i\beta) - (\alpha - i\beta))]$.

$$= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta)\tan(\alpha - i\beta)} = \frac{2i\sin\theta}{1+1} = i\sin\theta$$

$$\text{i}\tanh 2\beta = i\sin\theta$$

$$\therefore \tanh 2\beta = \sin\theta$$

$$\therefore 2\beta = \tanh^{-1}(\sin\theta) = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) .$$

But $1+\sin\theta = \left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2$.

But $1-\sin\theta = \left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)-2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2$.

$$\therefore 2\beta = \frac{1}{2}\log\left(\frac{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}\right)^2 = \log\left[\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right] .$$

$$\beta = \frac{1}{2}\log\left[\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right] = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) .$$

Q4)b) Expand $x^5+x^3-x^2+x-1$ **in powers of (x-1) and hence find the value of** **(6M)**

(1) $f\left(\frac{9}{10}\right)$

(2) $f(1.01)$

Ans : Let $f(x)=x^5+x^4-x^2+x-1$ and a=1 , $\therefore f(1)=1$.

$\therefore f'(x)=5x^4-4x^3+3x^2-2x+1,$ $\therefore f'(1)=3$.

$\therefore f''(x)=20x^3-12x^2+6x-2,$ $\therefore f''(1)=12$.

$\therefore f'''(x)=60x^2-24x+6,$ $\therefore f'''(1)=42$.

$\therefore f^{iv}(x)=120x-24,$ $\therefore f^{iv}(1)=96$.

$\therefore f^{v}(x)=120,$ $\therefore f^{iv}(1)=120$.

Now, $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+.......$.

$$\therefore f(x)=1+(x-1).3+\frac{(x-1)^2}{2!}.12+\frac{(x-1)^3}{3!}.42+\frac{(x-1)^4}{4!}.96+\frac{(x-1)^5}{5!}.120+....... .$$

$\therefore f(x)=1+(x-1).3+6(x-1)^2+7(x-1)^3+4(x-1)^4+(x-1)^5$.

(i) To find $f\left(\frac{9}{10}\right)$, we put x= 0.9 , and x-1 = -0.1

$\therefore f(0.9)=1+3(-0.1)+6(-0.1)^2+7(-0.1)^3+4(-0.1)^4+(-0.1)^5$

$=1-0.3+0.06-0.007+0.0004-0.00005$.

$=0.7534$

(ii) To find $f(1.01)$, we put x=1.01 and (x-1) =0.01 .

$\therefore f(1.01)=1+3(0.01)+6(0.01)^2+7(0.01)^3+4(0.01)^4+(0.01)^5$

$=1+0.03+0.0006-0.000007+0.00000004-0.0000000005$.

$=01.0306$

Q4)c) For what values of λ and μ , the equations , x + y + z =6 ; x + 2y +3z =10 ; x + 2y + λz =μ ;

(i) Have a unique solution

(ii) Have infinite solution

Find the solution in each case for a possible value of μ and λ . (8M)

Ans :

We have $\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$.

By $R_2 \to R_2 - R_1, R_3 \to R_3 - R_2$,

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & \lambda-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\4\\\mu-10\end{bmatrix}$$.

(i) The system has unique solution if the coefficient matrix is non-singular (or the rank A , r = the number of unknowns , n=3)

This requires $\lambda-3\neq 0, \quad \therefore \lambda \neq 3$.

$\therefore \lambda \neq 3$ then (μ may have any value)the system has unique solution .

(ii) If λ = 3 , the coefficient matrix and the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \mu-10 \end{bmatrix} .$$

The rank of A = 2 , and the rank of [A,B] will be also 2 if μ = 10 .

Thus if λ = 3 and μ = 10 , the system is consistent .But the rank of A(=2) is less than the number of unknowns (=3) . Hence the equations will possess infinite solutions .

Q5)a) Find the nth derivative of $y = \dfrac{1}{x^2+a^2}$. **(6M)**

Ans : We have

$$y = \frac{1}{x^2+a^2} = \frac{1}{x^2-a^2i^2} = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

$$y_n = \frac{1}{2ai}\left[\frac{(-1)^n.n!}{(x-ai)^{n+1}} - \frac{(-1)^n.n!}{(x+ai)^{n+1}}\right] .$$

$$= \frac{(-1)^n.n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$$

Let $x = r\cos\theta, a = r\sin\theta$, so that $r^2 = x^2+a^2, \theta = \tan^{-1}(\text{a/ x})$.

Now,

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} = \frac{1}{r^{n+1}}.\frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta}$$

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{r^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta]$$

$$\frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}} = \frac{1}{r^{n+1}}.\frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta} .$$

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{r^{n+1}}[\cos(n+1)\theta - i\sin(n+1)\theta]$$

$$\therefore \frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}}.2i\sin(n+1)\theta .$$

Putting these values in $\dfrac{(-1)^n.n!}{2ai}\left[\dfrac{1}{(x-ai)^{n+1}} - \dfrac{1}{(x+ai)^{n+1}}\right]$, we get

$$y_n = (-1)^n . n! . \frac{1}{a} . \frac{1}{r^{n+1}} \sin(n+1)\theta$$.

But r = $\frac{a}{\sin\theta}$. $\left(\because a = r\sin\theta, \therefore r^{n+1} = \frac{a^{n+1}}{\sin^{n+1}\theta} \right)$.

$$y_n = (-1)^n . n! \frac{1}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$$.

Q5)b) Discuss the maxima and minima of $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 16$. **(6M)**

Ans : We have f(x , y) = $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 16$.

Step I :

$$f_x = 3x^2 + y^2 - 24x + 21$$
$$f_y = 2xy - 4y$$
$$f_{xx} = 6x - 24, f_{xy} = 2y, f_{yy} = 2x - 4$$

Step II :

We now solve the equations $f_x = 0$, $f_y = 0$.

$\therefore 3x^2 + y^2 - 24x + 21 = 0$ and $2xy - 4y = 0$.

The second equation gives 2y(x-2) =0 .

$\therefore$ x = 2 or y = 0 .

When x=2 , the first equation $3x^2 + y^2 - 24x + 21 = 0$ gives

$\therefore 12 + y^2 - 48 + 21 = 0$ hence , y^2 -15 = 0 , y^2 =15 , y = $\pm\sqrt{15}$.

$\therefore$ The stationary values are (2, $\sqrt{15}$) , (2, - $\sqrt{15}$) .

When y = 0 , the first equation $3x^2 + y^2 - 24x + 21 = 0$ gives

$3x^2 - 24x + 21 = 0$, $x^2 - 8x + 7 = 0$.

(x-7) (x-1) = 0 , hence x = 1, 7 .

Therefore , the stationary values are (1,0) , (7,0) .

Step III :

(i) For x =2 , y = $\sqrt{15}$,

$r = f_{xx} = 12-24 = -12, s = f_{xy} = 2\sqrt{15}, t = f_{yy} = 4-4 = 0$.

$\therefore rt - s^2 = 0 - 60 = -60 < 0$.

$\therefore$ f(x,y) is neither maximum nor minimum. It is a saddle point .

(ii) For x =2 , y =- $\sqrt{15}$,

$r = f_{xx} = 12-24 = -12, s = f_{xy} = -2\sqrt{15}, t = f_{yy} = 4-4 = 0$.

$\therefore rt - s^2 = 0 - 60 = -60 < 0$.

$\therefore$ f(x,y) is neither maximum nor minimum. It is a saddle point .

(iii) For x =1 , y =0 ,

$r = f_{xx} = 6-24 = -18, s = f_{xy} = 0, t = f_{yy} = 2-4 = -2$

$\therefore rt - s^2 = 36 - 0 = 36 > 0$ and r = -18 , negative

$\therefore$ (1,0) is a maxima .

$\therefore$ The maximum value = 1 + 0 – 12 – 0 + 21 = 20 .

(iv) For x =7 , y =0 ,

$r = f_{xx} = 42-24 = 18, s = f_{xy} = 0, t = f_{yy} = 14-4 = 10$.

$\therefore rt - s^2 = 180 - 0 = 180 > 0$

Hence , (7,0) is a minima .

The minimum value = 343 + 0 -588- 0 +147 +10 =-88 .

Q5)c) Prove that if A and B are two unitary matrices then AB is also unitary. Verify the result when

$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ **and** $B = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ **.** **(8M)**

Ans : We have

$$(AB)(AB)^\theta = (AB)(B^\theta A^\theta) = A(BB^\theta)A^\theta$$.

$$= AIA^\theta \quad [\because B \text{ is unitary}]$$.

$$= AA^\theta = I \quad [\because A \text{ is unitary}]$$.

Similarly , we can prove that $(AB)^\theta(AB) = I$.

Hence , AB is also unitary .

Now, $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

$$\therefore A' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^{\theta} = (\bar{A}') = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore A' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^{\theta} = (\bar{A}') = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta} A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} . \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta} A = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now, $B = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$\therefore B' = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} \quad , \quad \therefore B^{\theta} = (\bar{B}') = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$B^{\theta} B = \frac{1}{4}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$B^{\theta} B = \frac{1}{4}\begin{bmatrix} (1-i^2)+(1-i^2) & -(1-i)^2+(1-i)^2 \\ -(1+i)^2+(1+i)^2 & (1-i)^2+(1-i)^2 \end{bmatrix}$$

$$B^{\theta} B = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence it is proved that if A and B are two unitary matrices then AB is also unitary and the result is verified .

Q6)a) If $x=\cosh\left(\frac{1}{m}\log y\right)$ **, prove that** $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$ **.** **(6M)**

Ans : $$x=\cosh\left(\frac{1}{m}\log y\right).$$

$$\cosh^{-1}x=\left(\frac{1}{m}\log y\right)$$

$$\log y=m\log\left(x+\sqrt{x^2-1}\right)=\log\left(x+\sqrt{x^2-1}\right)^m$$

Differentiating w.r.t x ,

$$y_1=m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right)$$

$$y_1=m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

$$=m\frac{\left(\sqrt{x^2-1}+x\right)^m}{\sqrt{x^2-1}}=\frac{my}{\sqrt{x^2-1}}$$

$$y_1\sqrt{x^2-1}=my$$
$$(x^2-1)y=m^2y^2$$

Differentiating w.r.t x ,

$$y_1\sqrt{x^2-1}=my$$
$$(x^2-1)2y_1y_2+2xy_1{}^2=m^2 2yy_1$$
$$(x^2-1)y_2+xy_1=m^2y$$

Differentiating n times using Leibnitz Theorem ,

$$(x^2-1)y_{n+2}+n.2xy_{n+1}+\frac{n(n-1)}{2!}2y_n+xy_{n+1}+ny_n=m^2y_n$$

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$$

Q6)b) Find a root of the equation $xe^x = \cos x$ using the Regula Falsi Method correct to three decimal places . (6M)

Ans :

$$f(x) = \cos x - xe^x = 0$$
$$f(0) = 1$$
$$f(1) = \cos 1 - e = -2.17798$$

The root lies between 0 and 1 .

Taking $x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -2.17798$,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{-1}{-2.17798 - 1} = \frac{1}{3.17798} = 0.3147$$

Now, $\cos 0.3147 - 0.3147e^{0.3147} = 0.5199$.

The value that we get is positive , so the root lies between 0.3147 and 1.

Taking $x_0 = 0.3147, x_1 = 1, f(x_0) = 0.5199, f(x_1) = -2.17798$,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.3147(2.1779) + 0.685335}{-2.17798 - (0.51987)} = \frac{1}{2.2384} = 0.44675$$

Now,

$$\cos 0.44675 - 0.44675e^{0.44675} = 0.2035$$

The value that we get is positive , so the root lies between 0.44675 and 1 .

Taking $x_0 = 0.44675, x_1 = 1, f(x_0) = 0.2035, f(x_1) = -2.17798$,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.44675(-2.1779) - 1 \text{x} \, 0.2035}{-2.17798 - (0.2035)} = \frac{1}{2.0242} = 0.494020$$

Now,

$$\cos 0.494020 - 0.494020e^{0.494020} = 0.0708$$

The value that we get is positive , so the root lies between 0.494020 and 1 .

Taking $x_0 = 0.494020, x_1 = 1, f(x_0) = 0.0708, f(x_1) = -2.17798$

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.494020(-2.1779) - 1\text{x}\,0.0708}{-2.17798 - (0.0708)} = \frac{1}{1.9316} = 0.51771 \quad .$$

Now,

$$\cos 0.51771 - 0.51771 e^{0.51771} = 0.00124 \quad .$$

Taking , $x_0 = 0.51771, x_1 = 1, f(x_0) = 0.00124, f(x_1) = -2.17798$.

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.51771(-2.1779) - 1\text{x}\,0.00124}{-2.17798 - (0.0708)} = \frac{1}{1.9315699} = 0.5177136 \quad .$$

Now, $\cos 0.5177136 - 0.5177136 e^{0.5177136} = 0.00124$.

If we compare x_2 and x_0 , we find that both are same upto four decimal places .

Hence , the root of the equation correct upto four decimal places is 0.5177 .

Q6)c)1) Expand $\sin^4\theta\cos^2\theta$ in a series of multiples of Θ . **(4M)**

Ans : Let $x = \cos\theta + i\sin\theta$ $\qquad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

Also $x^n + \frac{1}{x^n} = 2\cos n\theta$ and $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Now consider ,

$$(2i\sin\theta)^4(2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)^3 = \left(x^2 - \frac{1}{x^2}\right)^3\left(x - \frac{1}{x}\right)^3$$

$$= \left(x^6 - 3x^2 + 3.\frac{1}{x^2} - \frac{1}{x^6}\right)\left(x - \frac{1}{x}\right) \quad .$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$(2i\sin\theta)^4(2\cos\theta)^3 = 2\cos 7\theta - 2\cos 5\theta - 6\cos 3\theta + 6\cos\theta$$.

$$\sin^4\theta\cos^3\theta = \frac{\cos 7\theta}{2^6} - \frac{\cos 5\theta}{2^6} - \frac{3\cos 3\theta}{2^6} + \frac{3\cos\theta}{2^6}$$.

Q6)c)2) If one root of $x^4 - 6x^3 + 18x^2 - 24x + 16 = 0$ is (1+i) ; find the other roots. **(4M)**

Ans : Since (1+ i) is a root of the given equation , then we know that (1-i) must be one of the remaining roots because complex roots always occur in conjugate pairs . Hence , (x-1-i) and (x-1+i) are the factors of the left hand side , i.e the left hand side is divisible by

{ (x-1)-i } .{ (x-1)+i } , i.e. by $(x-1)^2 - i^2$ = $x^2 - 2x + 2$.

Dividing the left hand side by $x^2 - 2x + 2$, we get $x^2 - 8x + 32$.

Solving the equation $x^2 - 8x + 32$, we get $x^2 = 4 \pm 4i$.

Hence , the remaining roots are $(1-i),(4+4i),(4-4i)$.
