MUMBAI UNIVERSITY

SEMESTER 1

APPLIED MATHEMATICS SOLVED PAPER – MAY 2017

Q.1(a) Prove that $tanh^{-1}(\sin\theta) = cosh^{-1}(\sec\theta)$ **[3]**

Ans : L.H.S = $tanh^{-1}(\sin\theta)$

We know that, $tanh^{-1}(x) = \frac{1}{2}\log(\frac{1+x}{1-x})$

$\therefore$ L.H.S = $\frac{1}{2}\log(\frac{1+\sin\theta}{1-\sin\theta})$

R.H.S = $cosh^{-1}(\sec\theta)$

We know that, $cosh^{-1}(x) = \log(x+\sqrt{x^2-1})$

$\therefore$ R.H.S = $\log(\sec\theta+\sqrt{\sec^2\theta-1})$

$= \log(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta})$ ......$\{\sqrt{\sec^2\theta-1} = \tan\theta = \frac{\sin\theta}{\cos\theta}\}$

$= \log(\frac{1+\sin\theta}{\cos\theta})$

$= \log(\frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}})$

$= \log(\frac{\sqrt{1+\sin\theta}}{\sqrt{1-\sin\theta}})$

$= \frac{1}{2}\log(\frac{1+\sin\theta}{1-\sin\theta})$

$\therefore tanh^{-1}(\sin\theta) = cosh^{-1}(\sec\theta)$

Hence Proved .

(b) Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$ is unitary. [3]

Ans : Let A = $\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$

The matrix is unitary when A.$A^{\theta} = I$.

$$\therefore \quad A^{\theta} = (\bar{A})^t = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}^t = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore \quad A.A^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{\therefore \quad A.A^{\theta} = \ I}$$

The given matrix is unitary is proved.

(c) If x=uv & y=$\frac{u}{v}$ prove that $JJ^1 = 1$ [3]

Ans : x =uv and y=$\frac{u}{v}$

$\therefore$ x and y are function of u and v .

$\therefore$ u =$\sqrt{xy}$ $\therefore$ v=$\sqrt{\frac{x}{y}}$ .... { from given eqns }

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v} \quad ......(1)$$

$$J^1 = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix} = \frac{-\sqrt{\frac{x}{y}}}{2\sqrt{xy}} = \frac{-v}{2u} \quad ....(2)$$

$$\therefore JJ^1 = \frac{-2u}{v} \times \frac{-v}{2u} = 1$$

$$\boxed{\therefore JJ^1 = 1}$$

Hence Proved.

(d) If z =$\tan^{-1}(\frac{x}{y})$, where x=2t , y=1-t^2, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$. [3]

Ans : $z = \tan^{-1}(\frac{x}{y})$ $x=2t$ and $y=1-t^2$

∴ z is the function of x and y & x and y are the functions of t.

$z \longrightarrow f(x,y) \longrightarrow f(t)$

$\therefore z = \tan^{-1}(\frac{2t}{1-t^2})$

Direct differentiate w.r.t t ,

$$\frac{dz}{dt} = \frac{1}{1+(\frac{2t}{1-t^2})^2} \times \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$= \frac{2(1-t^2)^2}{(1-t^2)^2+4t^2} \times [t.\frac{1}{(1-t^2)^2}(-2t) + \frac{1}{1-t^2} \times 1]$$

$$= \frac{2(1-t^2)^2}{1+t^2} \times \frac{1}{(1-t^2)^2}$$

$$\therefore \frac{dz}{dt} = \frac{2}{1+t^2}$$

Hence Proved.

(e) Find the nth derivative of cos 5x.cos 3x.cos x. [4]

Ans : let y = cos 5x.cos 3x.cos x

$$= \frac{\cos(5x-3x)+\cos(5x+3x)}{2}. \cos x$$

$$= \frac{1}{2}[\cos 2x.\cos x + \cos 8x.\cos x]$$

$$y = \frac{1}{4}[\cos 3x + \cos x + \cos 9x + \cos 7x]$$

Take n th derivative,

$$n\ th\ derivative\ of\ \cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

$$y_n = \frac{1}{4}[9\cos\left(\frac{n\pi}{2}+3x\right) + \cos\left(\frac{n\pi}{2}+x\right) + 81\cos\left(\frac{n\pi}{2}+9x\right) + 49\cos\left(\frac{n\pi}{2}+7x\right)$$

(f) Evaluate : $\lim_{x\to 0}(x)^{\frac{1}{1-x}}$ [4]

Ans : Let $L = \lim_{x\to 0} (x)^{\frac{1}{1-x}}$

Take log on both the sides,

$$\therefore \log L = \lim_{x\to 0} \frac{\log x}{1-x}$$

Apply L'Hospital rule ,

$$\therefore \log L = \lim_{x\to 0} \frac{1}{x}$$

$$= 0$$

$$\boxed{\therefore\ L = e^0 = 1}$$

Q.2(a) Find all values of $(1+i)^{1/3}$ & show that their continued Product is (1+i). [6]

Ans : let $x = (1+i)^{1/3}$

$$\therefore\ x^3 = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$\therefore\ x^3 = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

Add period $2k\pi$,

$$x^3 = \sqrt{2}\left[\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right]$$

By applying De Moivres theorem,

$$x = 2\sqrt{2}\left[\cos\frac{1}{3}\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\frac{1}{3}\left(\frac{\pi}{4} + 2k\pi\right)\right]$$

where k =0,1,2.

Roots are :

Put k=0 $x_0 = 2\sqrt{2}e^{i\frac{\pi}{12}}$

Put k=1 $x_1 = 2\sqrt{2}e^{i\frac{9\pi}{12}}$

Put k=2 $x_2 = 2\sqrt{2}e^{i\frac{17\pi}{12}}$

The continued product of roots is given by ,

$$x_0x_1x_2 = 2\sqrt{2}e^{i\frac{\pi}{12}} \times 2\sqrt{2}e^{i\frac{9\pi}{12}} \times 2\sqrt{2}e^{i\frac{17\pi}{12}}$$

$$= 16\sqrt{2}\,e^{i\frac{27\pi}{12}}$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= 1+i$$

The continued product of roots is (1+i).

(b) Find non singular matrices P & Q such that PAQ is in normal form

Where $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ [6]

Ans : Matrix in PAQ form is given by ,

A = P A Q

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_3$,

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1, R_3 - 2R_1$,

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1, C_3 + C_1$,

$$\begin{bmatrix}1 & 0 & 0\\0 & -7 & 5\\0 & -6 & 5\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & -3\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -2 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$\frac{C_3}{5}$,

$$\begin{bmatrix}1 & 0 & 0\\0 & -7 & 1\\0 & -6 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & -3\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -2 & 1/5\\0 & 1 & 0\\0 & 0 & 1/5\end{bmatrix}$$

$C_2 + 6C_3$,

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & 1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & -3\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -4/5 & 1/5\\0 & 1 & 0\\0 & 6/5 & 1/5\end{bmatrix}$$

$C_3 + C_2$,

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & -3\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -4/5 & -3/5\\0 & 1 & 1\\0 & 6/5 & 7/5\end{bmatrix}$$

- R_1,

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & -1 & 3\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -4/5 & -3/5\\0 & 1 & 1\\0 & 6/5 & 7/5\end{bmatrix}$$

Now A is in normal form with rank 3.

Compare with PAQ form ,

$$P = \begin{bmatrix}0 & 0 & 1\\0 & -1 & 3\\1 & 0 & -2\end{bmatrix} \qquad Q = \begin{bmatrix}1 & -4/5 & -3/5\\0 & 1 & 1\\0 & 6/5 & 7/5\end{bmatrix}$$

(c) Find the maximum and minimum values of

$$f(x,y)=x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \qquad [8]$$

Ans : given : $f(x,y)=x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 - 30x + 72 \quad f_{xx} = 6x - 30$

$f_y = 6xy - 30y \quad f_{yy} = 6x - 30$

$f_{xy} = 6y$

To find stationary values :

$f_x = 3x^2 + 3y^2 - 30x + 72 = 0 \quad \& \ f_y = 6xy - 30y = 0$

y=0 or x=5

for y=0 , x=6,4

∴ (x,y)=(6,0) , (4,0).

For x=5 , y=1,-1

∴ (x,y)=(5,1) , (5,-1)

Stationary points are : (6,0),(4,0),(5,1),(5,-1)

(i) For point (6,0) ,

r = f_{xx}=36-30=6 , s =$f_{xy} = 0$, $t = f_{yy} = 6$

rt - $s^2 = 36 > 0$ and r =6>0

function is minimum at (6,0).

$f_{min} = 108$

(ii) For point (4,0) ,

r = f_{xx}=-6 , s =$f_{xy} = 0$, $t = f_{yy} = -6$

rt - $s^2 = 36 > 0$ and r =-6<0

function is maximum at (4,0).

$f_{max} = 112$

(iii) for point (5,1) and (5,-1) ,

Thr points are neither maximum nor minimum.

∴ The maximum and minimum value of function are 112 and 108 .

Q.3(a) If u = f($\frac{y-x}{xy}$, $\frac{z-x}{xz}$) , show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$ [6]

Ans : let u = f(r,s)

$\therefore$ r = $\frac{y-x}{xy}$ $\therefore$ s = $\frac{z-x}{xz}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\frac{1}{x^2} + \frac{\partial u}{\partial s}(\frac{-1}{x^2})$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial u}{\partial r}\frac{(-1)}{y^2} + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(\frac{1}{z^2})$$

$$\therefore \quad x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\boxed{x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0}$$

Hence proved.

(b) Using encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,encode & decode the message "MUMBAI". [6]

Ans : Encoding matrix : A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Message is : MUMBAI.

The given message in matrix form is :

$$B = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

Encoded message in matrix form is given by ,

$$C = A.B$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

$$\therefore \; C = \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix}$$

Encoded message is : 34 21 15 2 10 9

GUOBJI

Decoded matrix is given by ,

$$B = A^{-1}.C$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

Decoded message : MUMBAI

(c) Prove that $\log[\tan(\frac{\pi}{4} + \frac{ix}{2})] = i.\tan^{-1}(sinhx)$ [8]

Ans : L.H.S $= \log[\tan(\frac{\pi}{4} + \frac{ix}{2})]$

$$= \log\left[\frac{1+tan\,(\frac{ix}{2})}{1-tan\,(\frac{ix}{2})}\right]$$

$$= \log[1+ tan\,(\frac{ix}{2})] - \log[1 - tan\,(\frac{ix}{2})]$$

$$= \log[1+i.\tanh\frac{x}{2}] - \log[1-i\tanh\frac{x}{2}]$$

We have ,

$$\log(a+ib) = \frac{1}{2}log(a^2 + b^2) + i\tan^{-1}(\frac{b}{a})$$

$$\therefore = \frac{1}{2}log\left(1 + tanh^2\frac{x}{2}\right) + i\tan^{-1}\left(tanh\frac{x}{2}\right) - [\frac{1}{2}log\left(1 + tanh^2\frac{x}{2}\right) - i\tan^{-1}\left(tanh\frac{x}{2}\right)]$$

$$= 2i[\tan^{-1}\left(tanh\frac{x}{2}\right)]$$

$$\text{L.H.S} = 2i.\tan^{-1}\left(tanh\frac{x}{2}\right)$$

$$\text{R.H.S} = i.\tan^{-1}(sinhx)$$

We know that $sinh^{-1}x = log(x + \sqrt{1 + x^2})$

$$tanh^{-1}x = \frac{1}{2}[log(\frac{x+1}{1-x})]$$

$= \text{i}\tan^{-1}\left(tanh\frac{x}{2}\right)$

Also $sinh^{-1}(tanx) = tanh^{-1}(x)$

R.H.S $=$ i. $\tan^{-1}\left(tanh\frac{x}{2}\right)$

$$\log[\tan(\frac{\pi}{4}+\frac{ix}{2})]=\text{i}.\tan^{-1}(\,sinhx)$$

Q.4(a) Obtain tan 5θ in terms of tan θ & show that

$$1\text{-}10tan^2\frac{x}{10} + 5tan^4\frac{x}{10} = 0 \qquad [6]$$

Ans : we have $\tan 5\theta = \frac{sin\ 5\theta}{cos\ 5\theta}$

$$(\cos\theta + i\sin\theta)^n = cos\ n\theta + i\ sin\ n\theta$$

Put n=5 ,

$$\therefore\ cos\ 5\theta + i\ sin\ 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= cos^5\theta + 5cos^4\theta.isin\theta + 10cos^3\theta.(isin\theta)^2$$

$$+10cos^2\theta.(isin\theta)^3 + 5cos\theta.(isin\theta)^4 + isin^5\theta$$

$$= [cos^5\theta - \ 10cos^3\theta.(sin\theta)^2$$

$$+ 5cos\theta.(sin\theta)^4\ ] + [5cos^4\theta.isin\theta$$

$$- 10icos^2\theta.(sin\theta)^3 + isin^5\theta]$$

Compare real and imaginary parts

$$\cos 5\theta = [cos^5\theta - \ 10cos^3\theta.(sin\theta)^2 + \ 5cos\theta.(sin\theta)^4$$

$$\sin 5\theta = +[5cos^4\theta.sin\theta - 10cos^2\theta.(sin\theta)^3 + sin^5\theta\ ]$$

$$\tan 5\theta = \frac{[5cos^4\theta.sin\theta - 10cos^2\theta.(sin\theta)^3 + sin^5\theta\ ]}{[cos^5\theta - \ 10cos^3\theta.(sin\theta)^2 + \ 5cos\theta.(sin\theta)^4}$$

$$\tan 5\theta = \frac{5tan\theta - 10tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta}$$

put $\theta = \frac{\pi}{10}$

$$1\text{-}10\tan^2\frac{x}{10} + 5\tan^4\frac{x}{10} = 0$$

(b) If $y=e^{\tan^{-1}x}$. Prove that

$$(1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0$$ [6]

Ans : $y = e^{\tan^{-1}x}$ ....(1)

Diff. w.r.t x ,

$$y_1 = e^{\tan^{-1}x}\frac{1}{x^2+1}$$

$$(x^2+1)y_1 = e^{\tan^{-1}x} = y$$ --------(from 1)

Again diff. w.r.t x,

$$(x^2+1)y_2 + 2xy_1 = y_1$$ ....(1)

Now take n th derivative by applying Leibnitz theorem,

Leibnitz theorem is :

$$(uv)_n = u_n v + {}^n_1C u_{n-1}v_1 + {}^n_2C u_{n-2}v_2 + \cdots + uv_n$$

$u = (x^2+1), v = y_2$ ...for first term in eqn (1)

$u = 2x , v = y_1$ ...for second term in eqn (1)

$$\therefore\ (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n - y_{n+1} = 0$$

$$\therefore\ (1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0$$

Hence Proved.

(c) i. Express $(2x^3+3x^2-8x+7)$ in terms of (x-2) using Taylor's Series. [4]

ii. Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$ [4]

Ans : i. let f(x)= $2x^3 + 3x^2 - 8x + 7$

Here a = 2

$f(x) = 2x^3 + 3x^2 - 8x + 7$ $f(2) = 19$

$f'(x) = 6x^2 + 6x - 8$ $f'(2) = 28$

$f''(x) = 12x + 6$ $f''(2) = 30$

$f'''(x) = f'''(2) = 12$

Taylor's series is :

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$2x^3 + 3x^2 - 8x + 7 = 19 + (x-2)28 + \frac{(x-2)^2}{2!} 30 + \frac{(x-a)^3}{3!} 12$$

$$\boxed{2x^3 + 3x^2 - 8x + 7 = 19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3}$$

ii. let $y = \tan^{-1} x$

diff. w.r.t x ,

$\therefore$ $y_1 = \frac{1}{x^2+1}$

Series expansion of y_1 ,

We know that ,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$\therefore$ $y_1 = 1 - x^2 + x^4 - x^5$

Integrate y_1 to find series expansion of y,

$\therefore$ $y = \int (1 - x^2 + x^4 - x^5 + \cdots) dx$

$$\therefore \; y = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

Hence Proved .

Q.5(a) If $z = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y} \quad \partial$

Prove that $\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$ [6]

Ans : $z = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Diff. w.r.t. x partially ,

$$\frac{\partial z}{\partial x} = x^2 \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2} + \tan^{-1}\frac{y}{x} \,.\, 2x - y^2 \frac{y^2}{x^2+y^2} \times \frac{1}{y}$$

$$= \frac{x^2}{x^2+y^2} \times \frac{-y}{1} + 2x\tan^{-1}\frac{x}{y} - \frac{y^3}{x^2+y^2}$$

Diff. w.r.t y partially ,

$$\frac{\partial^2 z}{\partial y \partial x} = -x^2\left[-y.\frac{2y}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}\right] + 2\,\frac{x^2}{x^2+y^2} - \left[-y^3.\frac{2y}{(x^2+y^2)^2} + \frac{3y^2}{x^2+y^2}\right]$$

$$= \left[.\frac{2y^3x^2}{(x^2+y^2)^2} + \frac{-x^2}{x^2+y^2}\right] + 2\,\frac{x^2}{x^2+y^2} + \frac{2y^4}{(x^2+y^2)^2} - \frac{3y^2}{x^2+y^2}$$

$$= \frac{(x^2-y^2)^2 \times}{(x^2+y^2)^2}\frac{(x^2+y^2)^1}{(x^2-y^2)^1}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$$

Hence proved.

(b) Investigate for what values of μ and λ the equations :2x+3y+5z=9

7x+3y-2z =8

2x+3y+λz=μ

Have (i) no solution (ii) unique solution (iii) Infinite value [6]

Ans : **Given eqn :** 2x+3y+5z=9

7x+3y-2z =8

2x+3y+λz=μ

A X = B

$$\therefore \quad \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Argumented matrix is : $\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$

$R_3 - R_1$,

$$\rightarrow \quad \left[\begin{array}{ccc|c} 2 & 3 & 5 & 6 \\ 7 & 3 & -2 & 4 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

(i) When λ=5,$\mu \neq 9$ *then* $r(a) = 2, r(A \vdots B) = 3$

r(A)$\neq r(A \vdots B)$

No Solution.

(ii) When λ≠5,$\mu \neq 9$, $r(A) = r(A \vdots B) = 3$

Unique solution exist.

(iii) When λ=5,$\mu = 9$ $\quad r(A) = r(A \vdots B) = 2 < 3$

Infinite solution.

(c) Obtain the root of $x^3 - x - 1 = 0$ by Newton Raphson Method (upto three decimal places). **[8]**

Ans : **Equation :** $x^3 - 2x - 5 = 0$

$$\therefore \; f(x) = x^3 - 2x - 5$$

$$f(0) = -5 < 0 \;\; and \;\; f(1) = -2 < 0 \; and \;\; f(2) = 7 > 0.$$

Root of given eqn lies between 1 and 2.

$$f'(x) = 3x^2 + 2$$

Let take $x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{7}{14} = 1.5$$

Next iteration :

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.343$$

$$\therefore \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.329$$

For next iteration :

$$\therefore \quad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.329 - \frac{f(1.329)}{f'(1.329)}$$

$$= 1.3283$$

The root of eqn is x = 1.3283

Q.6(a) Find tanhx if 5sinhx-coshx = 5 **[6]**

Ans : 5sinhx-coshx = 5

But sinhx = $\frac{e^x - e^{-x}}{2}$ coshx = $\frac{e^x + e^{-x}}{2}$

$$\therefore \; 5\left[\frac{e^x - e^{-x}}{2}\right] - \left[\frac{e^x + e^{-x}}{2}\right] = 5$$

$$\therefore \; 5e^x - 5e^{-x} - e^x - e^{-x} = 10$$

$$\therefore \; 4e^{2x} - 10e^x - 6 = 0$$

Roots are : $e^x = 3$, $e^x = \frac{-1}{2}$

$\therefore$ tanhx = $\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\left(-\frac{1}{2}\right)+2}{-5/2} = \frac{-3}{5}$

Or

$\therefore$ tanhx = $\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{3-1/3}{3+1/3} = \frac{4}{5}$

The values of tanhx are : $\frac{-3}{5}$ *or* $\frac{4}{5}$

(b) If u = $\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *Prove that* i. $xu_x + yu_y = \frac{1}{2}$ tanu

ii. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-sinu.cos2u}{4cos^3u}$ [6]

Ans : u = $\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Put x = xt and y = yt to find degree.

$\therefore$ u=$\sin^{-1}\left(\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}\right)$

$\therefore$ sin u = $t^{1/2}.\frac{x+y}{\sqrt{x}+\sqrt{y}} = t^{\frac{1}{2}}.f(x,y)$

The function sin u is homogeneous with degree ½.

But sin u is the function of u and u is the function of x and y.

By Euler's theorem ,

$$xu_x + yu_y = G(u) = n.\frac{f(u)}{f'(u)} = \frac{1}{2}\text{tanu}$$

$$\therefore xu_x + yu_y = \frac{1}{2}\text{tanu}$$

$$\therefore x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u)[G'(u) - 1]$$

$$= \frac{1}{2}\tan u\left[\frac{sec^2u-2}{2}\right]$$

$$= \frac{1}{4}\tan u[\frac{tan^2u-1}{1}]$$

$$= \frac{1}{4} \times \frac{\sin u}{\cos u}[\frac{sin^2u-cos^2u}{cos^2u}]$$

$$\therefore \quad x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-sinu.cos2u}{4cos^3u}$$

Hence Proved.

(c) Solve the following system of equation by Gauss Siedal Method,

20x+y-2z=17

3x+20y-z =-18

2x-3y+20z=25 [8]

Ans : **By Gauss Seidal method ,**

Given eqn : 20x+y-2z=17

3x+20y-z =-18

2x-3y+20z=25

From given eqn : |20|>|1|+|-2|

|20|>|3|+|-1|

|20|>|2|+|-3|

The given eqn are in correct order.

$$\therefore \ x = \frac{1}{20}[17 - y + 2z]$$

$$\therefore y = \frac{1}{20}[-18 - 3x + z]$$

$$\therefore \ z = \frac{1}{20}[25 - 2x + 3y]$$

I) For 1st iteration : take $y = 0, z = 0$

$$x = \frac{1}{20}[17] = 0.85$$

$x = 0.85\,, z = 0$ gives $y = -1.0275$

$x = 0.85, y = -1.0275$ gives $x_3 = 1.0109$

II) **For 2nd iteration : take** $y = -1.0275, z = 1.0109$

$x = \frac{1}{20}[17 + 1.0275 - 2(1.0109)] = 1.0025$

$x = 1.0025, z = 1.0109$ gives $y = -0.9998$

$x = 1.0025, y = -0.9998$ gives $z = 0.9998$

III) **For 3rd iteration :** $y = -0.9998, z = 0.9998$

$x_1 = \frac{1}{20}[17 + 0.9998 + 2(0.9998)] = 1.00$

$x = 1.00, z = 0.9998$ gives $y = -1.00$

$x = 1.00, y = -1.00$ gives $z = 1.00$

Result : $x = 1.00, y = -1.00, z = 1.00$