APPLIED MATHEMATICS-1

Dec 2017

Mumbai University Dec 2017 (CBCS) solutions.

Q.1) Answer the following **(20 marks)**

1a) Separate into real and imaginary parts of cos$^{-1}\left(\frac{3i}{4}\right)$. **(3 marks)**

Ans. Let $a + ib = \cos^{-1}\left(\frac{3i}{4}\right)$ ....(1)

$\therefore \cos(a + ib) = \frac{3i}{4}$

$\therefore \cos(a)\cos(ib) - \sin(a)\sin(ib) = \frac{3i}{4}$

$\cos(a)\cosh(b) - i\sin(a)\sinh(b) = 0+\frac{3i}{4}$ $\{\because \cos(ix)=\cosh(x)\text{ , }\sin(ix)=\sinh(x);\}$

Comparing Real and Imaginary terms on both sides,

$\cos(a)\cosh(b)=0$ ....(2) & $-\sin(a)\text{sinj}(b) = \frac{3}{4}$ ....(3)

From (2), $\cos(a)=0$ or $\cosh(b)=0$,

$\therefore$ $a = \frac{\pi}{2}$ ....(4)

From (3) & (4), $-\sin\left(\frac{\pi}{2}\right)\sinh(b) = \frac{3}{4}$

$\therefore$ $1.\sinh(b) = \frac{-3}{4}$

$\therefore$ $b = \sinh^{-1}\left(\frac{-3}{4}\right)$

$$= \log\left[\left(\frac{-3}{4}\right) + \sqrt{\left(\frac{-3}{4}\right)^2 + 1}\right] \quad \{\because sinh^{-1}z = log\left(z + \sqrt{z^2+1}\right)\}$$

$$=\log\left[\left(\frac{-3}{4}\right) + \sqrt{\frac{9}{16} + 1}\right]$$

$=\log\left[\left(\frac{-3}{4}\right)+\frac{5}{4}\right]$

$=\log\frac{1}{2}$

$=\log 2^{-1}$

$\therefore b = -\log 2$ ....(5)

Substituting (4) & (5) in (1), $\cos^{-1}\left(\frac{3i}{4}\right) = \frac{\pi}{2} - i\log 2$

Comparing Real and Imaginary terms on both sides,

Real part = $a = \frac{\pi}{2}$

Imaginary part = $b = -\log 2$

1b) Show that the matrix A is unitary where A = $\begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$ **is unitary if**

$\alpha^2+\beta^2+\gamma^2+\delta^2 = 1$ **(3 marks)**

Ans: $A=\begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$

$\therefore A^T=\begin{bmatrix} \alpha+i\gamma & \beta+i\delta \\ -\beta+i\delta & \alpha-i\gamma \end{bmatrix}$

$\therefore A^\theta=\overline{A^T}=\begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix}$

Given, A is Unitary

$\therefore AA^\theta=1$

$\therefore \begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix} \times \begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \begin{bmatrix} (\alpha+i\gamma)(\alpha-i\gamma)+(-\beta+i\delta)(-\beta-i\delta) & (\alpha+i\gamma)(\beta-i\delta)+(-\beta+i\delta)(\alpha+i\gamma) \\ (\beta+i\delta)(\alpha-i\gamma)+(\alpha-i\gamma)(-\beta-i\delta) & (\beta+i\delta)(\beta-i\delta)+(\alpha-i\gamma)(\alpha+i\gamma) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ...(1)

Consider,

$(\alpha + i\gamma)(\alpha - i\gamma) + (-\beta + i\delta)(-\beta - i\delta) = \alpha^2 - i^2\gamma^2 + \beta^2 - i^2\delta^2$

$(\alpha + i\gamma)(\alpha - i\gamma) + (-\beta + i\delta)(-\beta - i\delta) = \alpha^2+\beta^2+\gamma^2+\delta^2$ ...(2)

And, $(\alpha + i\gamma)(\beta - i\delta) + (-\beta + i\delta)(\alpha + i\gamma) = (\alpha\beta - i\alpha\delta + i\beta\gamma - i^2\gamma\delta) + (-\alpha\beta - i\beta\gamma + i\alpha\delta + i^2\gamma\delta) = 0$ ...(3)

Similarlly,

$(\beta + i\delta)(\alpha - i\gamma) + (\alpha - i\gamma)(-\beta - i\delta) = 0$ ...(4)

$(\beta + i\delta)(\beta - i\delta) + (\alpha - i\gamma)(\alpha + i\gamma) = \alpha^2+\beta^2+\gamma^2+\delta^2$ ...(5)

Substituting (2),(3),(4)&(5) in (1),

$$\begin{bmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing corresponding terms, we get,

$$\boxed{\alpha^2+\beta^2+\gamma^2+\delta^2=1}$$

1c) If z = tan(y + ax) + (y - ax)$^{3/2}$ then show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$ (3 marks)

Ans: $z = \tan(y + ax) + (y - ax)^{3/2}$ ...(1)

Differentiate partially w.r.t.x, $\frac{\partial z}{\partial x} = \sec^2(y + ax)\cdot a + \frac{3}{2}(y - ax)^{1/2}\cdot(-a)$

$\therefore \frac{\partial z}{\partial x} = a\sec^2(y + ax) - \frac{3a}{2}(y - ax)^{1/2}$

Again, differentiate partially w.r.t.x,

$\frac{\partial^2 z}{\partial x^2} = 2a^2\sec^2(y + ax)\cdot \tan(y + ax) - \frac{3}{4}a^2(y - ax)^{-1/2}$

$\therefore \frac{\partial^2 z}{\partial x^2} = a^2\left[2sec^2(y + ax)\tan(y + ax) - \frac{3}{4}(y - ax)^2\right]$ ...(2)

Differentiate (1) partially w.r.t.y, $\frac{\partial z}{\partial y} = \sec^2(y + ax)\cdot 1 + \frac{3}{2}(y - ax)^{1/2}$

Again, differentiate partially w.r.t.y, $\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y + ax)\cdot \tan(y + ax) - \frac{3}{4}(y - ax)^{-1/2}$ ...(3)

From (2)&(3), $\boxed{\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}}$

1d) If $x=uv$, $y=\frac{u}{v}$. Prove that $JJ'=1$. (3marks)

Ans: $x=uv$ ...(1)

$\therefore x_u = \frac{\partial x}{\partial u} = v$ and $x_v = \frac{\partial x}{\partial v} = u$ ...(2)

And, $y = \frac{u}{v}$ ...(3)

$\therefore y_u = \frac{\partial y}{\partial u} = \frac{1}{v}$ and $y_v = \frac{\partial y}{\partial v} = u\frac{-1}{v^2}$ ...(4)

$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$= x_u y_v - x_v y_u$

$= v\, u \frac{-1}{v^2} - u\frac{1}{v}$ ...(From 2 & 4)

$= \frac{-u}{v} - \frac{u}{v}$

$= \frac{-2u}{v}$

$\therefore J = -2y$ ...(5)

From (3), $u = vy$ ...(6)

Substituting 'u' in (1) we get, $x= (vy)v$

$\frac{x}{y} = v^2$

$\therefore v = \frac{\sqrt{x}}{\sqrt{y}} = x^{1/2}y^{1/2}$ ...(7)

$\therefore v_x = y^{-1/2}.\frac{1}{2} x^{-1/2}$ and $v_y = x^{1/2}.\frac{-1}{2} y^{-3/2}$ ...(8)

From (6) and (7), $u = \left(x^{1/2}y^{-1/2}\right)y$

$\therefore u = x^{1/2}y^{1/2}$

$\therefore$ $u_x = y^{1/2}$. and $u_y = x^{1/2}.\frac{1}{2}\, y^{-1/2}$ ...(9)

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= u_x v_y - u_y v_x$$

$$= \left(y^{1/2} \cdot \frac{1}{2}\, x^{-1/2}\right)\left(x^{1/2} \cdot \frac{-1}{2}\, y^{-3/2}\right) - \left(x^{1/2} \cdot \frac{1}{2}\, y^{-1/2}\right)\left(y^{-1/2} \cdot \frac{1}{2}\, x^{-1/2}\right)$$ ...(From 8 & 9)

$$= \frac{-1}{4}\, x^{\frac{-1}{2}+\frac{1}{2}} \cdot y^{\frac{1}{2}-\frac{3}{2}} - \frac{-1}{4}\, x^{\frac{1}{2}-\frac{1}{2}} \cdot y^{\frac{-1}{2}-\frac{1}{2}}$$

$$= \frac{-1}{4} \cdot y^{-1} - \frac{1}{4} \cdot y^{-1}$$

$$= \frac{-2}{4} \cdot y^{-1}$$

$$\therefore J' = \frac{-1}{2y}$$ ...(10)

From (5) and (10), $J \cdot J' = -2y \cdot \frac{-1}{2y}$

$$\boxed{\therefore J \cdot J' = 1}$$

1e) Find the n

1e) Find the n^{th} derivative of $\frac{x^3}{(x+1)(x-2)}$. **(4 marks)**

Ans: Let $y = \frac{x^3}{(x+1)(x-2)} = \frac{x^3}{x^2-x-2}$

Consider, $x^2 - x - 2 \overline{)\, x^3 + 0x^2 + 0x + 0}$ with quotient $x+1$

$$\underline{x^3 - x^2 - 2x}$$

$$x^2 + 2x + 0$$

$$\underline{x^2 - x - 2}$$

$$3x + 2$$

$$\therefore y = x + 1 + \frac{3x+2}{x^2-x-2}$$

$\therefore$ y = x + 1 + $\frac{3x+2}{(x+1)(x-2)}$

$\therefore$ y = x + 1 + $\frac{1/3}{(x+1)}+\frac{8/3}{(x-2)}$ (By Partial Fraction)

Taking nth order derivative, $y_n = 0+0+\frac{1}{3}\cdot\frac{n!\cdot 1^n\,(-1)^n}{(x+1)^{n+1}}+\frac{8}{3}\cdot\frac{n!\cdot 1^n\,(-1)^n}{(x-2)^{n+1}}$

$$\left\{\text{If y} = \frac{1}{ax+b}\ then\ y_n = \frac{n!a^n(-1)^n}{(ax+b)^{n+1}}\right\}$$

$$\boxed{\therefore y_n = \frac{n!\,(-1)^n}{3}\left[\frac{1}{(x+1)^{n+1}}+\frac{8}{(x-2)^{n+1}}\right]}$$

1f) Using the matrix A = $\begin{bmatrix}-1 & 2\\-1 & 1\end{bmatrix}$ decode the message of matrix C = $\begin{bmatrix}4 & 11 & 12 & -2\\-4 & 4 & 9 & -2\end{bmatrix}$

(4 marks)

Ans: Encoding Matrix A = $\begin{bmatrix}-1 & 2\\-1 & 1\end{bmatrix}$ ...(1)

Given, C = $\begin{bmatrix}4 & 11 & 12 & -2\\-4 & 4 & 9 & -2\end{bmatrix}$

Step 1:

Writing the numbers in C matrix column wise gives the encoded message.

$\therefore$ Encoded Message = 4 -4 11 4 12 9 -2 -2

This Encoded message is transmitted.

Assume there is no corruption of data, the message at the receiving end is 4 -4 11 4 12 9 -2 -2

This message is decoded

Step 2;

We know, if P = $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ then $P^{-1} = \frac{1}{|P|}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

From (1), |A| = -1 + 2 = 1 ...(2)

$\therefore$ Decoding matrix $A^{-1} = \frac{1}{1}\begin{bmatrix}1 & -2\\1 & -1\end{bmatrix}$ (From 2) ...(3)

From (2) & (3), $A^{-1}C = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \text{ x } \begin{bmatrix} 4 & 11 & 12 & -2 \\ -4 & 4 & 9 & -2 \end{bmatrix}$

$\therefore A^{-1}C = \begin{bmatrix} 4+8 & 11-8 & 12-18 & -2+4 \\ 4+4 & 11-4 & 12-9 & -2+2 \end{bmatrix}$

$\therefore A^{-1}C = \begin{bmatrix} 12 & 3 & 6 & 2 \\ 8 & 7 & 3 & 0 \end{bmatrix}$

Step 3:

Considering the numbers column-wise we get,

12 8 3 7 -6 3 2 0

$\boxed{\text{Decoded Message = 12 8 3 7 -6 3 2 0 or } \begin{bmatrix} 12 & 3 & 6 & 2 \\ 8 & 7 & 3 & 0 \end{bmatrix}}$

Q.2) (20 marks)

2a) If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$ then find a,b,c,d. (6 marks)

Ans: We know, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ ...(1)

Consider, $\sin^4\theta\cos^3\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \text{ x } \frac{e^{i\theta}+e^{-i\theta}}{2}$ (From 1)

$$= \frac{1}{2^4 i^4 2^3} \text{ x } \left(e^{i\theta} - e^{-i\theta}\right)\left(e^{i\theta} - e^{-i\theta}\right)^3\left(e^{i\theta} + e^{-i\theta}\right)^3$$

$$= \frac{1}{2^7} \text{ x } \left(e^{i\theta} - e^{-i\theta}\right)\left[\left(e^{i\theta}\right)^2 - \left(e^{-i\theta}\right)^2\right]^3$$

$$= \frac{1}{2^7} \text{ x } \left(e^{i\theta} - e^{-i\theta}\right)\left[\left(e^{2i\theta}\right) - \left(e^{-2i\theta}\right)\right]^3$$

$$= \frac{1}{2^7} \text{ x } \left(e^{i\theta} - e^{-i\theta}\right)\left[\left(e^{2i\theta}\right)^3 - 3\left(e^{2i\theta}\right)^2\left(e^{-2i\theta}\right) + 3\left(e^{2i\theta}\right)\left(e^{-2i\theta}\right)^2 - \left(e^{-2i\theta}\right)^3\right]$$

$$= \frac{1}{2^7} \text{ x } \left(e^{i\theta} - e^{-i\theta}\right)\left[e^{6i\theta} - 3e^{2i\theta} + 3e^{-2i\theta} - e^{-6i\theta}\right]$$

$$= \frac{1}{2^7} \text{ x } \left[e^{7i\theta} - 3e^{3i\theta} + 3e^{-i\theta} - e^{-5i\theta} - e^{5i\theta} + 3e^{i\theta} - 3e^{-3i\theta} + e^{-7i\theta}\right]$$

$$= \frac{1}{2^7}\left[\left(e^{7i\theta} + e^{-7i\theta}\right) - \left(e^{5i\theta} + e^{-5i\theta}\right) - 3\left(e^{3i\theta} + e^{-3i\theta}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right)\right]$$

$$= \frac{1}{128} \text{ x } 2\cos7\theta - \frac{1}{128} \text{ x } 2\cos5\theta - \frac{1}{128} \text{ x } 6\cos3\theta + \frac{1}{128} \text{ x } 6\cos\theta$$

$\therefore \sin^4\theta\cos^3\theta = \frac{3}{64}\cos\theta - \frac{3}{64}\cos3\theta - \frac{1}{64}\cos5\theta + \frac{1}{64}\cos7\theta$ ...(2)

But , given, $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$ ...(3)

Comparing (2) & (3), $\boxed{a = \frac{3}{64}\,;\ b = \frac{-3}{64}\,;\ c = \frac{-1}{64}\,;\ d = \frac{1}{64}\,;}$

2b) Using Newton Raphson method solve 3x – cosx – 1 = 0. Correct upto 3 decimal places.

(6 marks)

Ans: Let $f(x) = 3x - \cos x - 1$

$\therefore f'(x) = 3 + \sin x - 0$

When x = 0, $f(0) = 3(0) - \cos0 - 1 = -2$

When x =1, $f(1) = 3(1) - \cos1 - 1 = 1.4597$

$\therefore$ Roots of $f(x)$ lies between 0 and 1.

Let initial value $x_0 = 0$

By Newton-Raphson's Method $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$= x_n - \frac{3x_n - cosx_n - 1}{3 + sinx_n}$

$= \frac{x_n(3+sinx_n) - (3x_n - cosx_n - 1)}{3+sinx_n}$

$= \frac{3x_n + x_n sinx_n - 3x_n + cosx_n + 1}{3+sinx_n}$

$\therefore x_{n+1} = \frac{x_n sinx_n + cosx_n + 1}{3+sinx_n}$ ...(1)

Iteration 1: Put n = 0 in (1)

$\therefore x_1 = \frac{x_0 sinx_0 + cosx_0 + 1}{3+sinx_0} = \frac{0+cos0+1}{3+sin0} = 0.6667$

Iteration 2: Put n = 1 in (1)

$\therefore x_2 = \frac{x_1 sinx_1 + cosx_1 + 1}{3+sinx_1} = \frac{0.6667sin(0.6667)+cos(0.6667)+1}{3+sin(0.6667)} = 0.6075$

Iteration 3: Put n = 2 in (1)

$\therefore$ $x_3 = \frac{x_2 sinx_2 + cosx_2 + 1}{3 + sinx_2} = \frac{0.6075 sin(0.6075) + cos(0.6075) + 1}{3 + sin(0.6075)}$ = 0.6071

Iteration 4: Put n = 3 in (1)

$\therefore$ $x_4 = \frac{x_3 sinx_3 + cosx_3 + 1}{3 + sinx_3} = \frac{0.6071 sin(0.6071) + cos(0.6071) + 1}{3 + sin(0.6071)}$ = 0.6071

Hence, Root of 3x – cosx – 1 = 0 is 0.6071

2c) Find the stationary points of the function $x^3+3xy^2-3x^2-3y^2+4$ & also find maximum and minimum values of the function. (8 marks)

Ans: Let $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ ...(1)

$\therefore f_x = 3x^2 + 3y^2 - 6x - 0 + 0$

$\therefore$ r = $f_{xx} = 6x - 6$ ...(2)

Also, $f_y = 0 + 6xy - 0 - 6y + 0$

$\therefore$ t = $f_{yy} = 6x - 6$ ...(3)

$\therefore$ s = $f_{xy} = 0 + 6y - 0$ ...(4)

Put $f_x = 0$ and $f_y = 0$

$\therefore 3x^2 + 3y^2 - 6x = 0$

$\therefore x^2 + y^2 - 2x = 0$ ...(5)

And, $6xy - 6y = 0$

$\therefore 6y(x-1) = 0$

$\therefore$ y = 0 or x = 1

Case I : When x = 1

From (5), $1^2 + y^2 - 2(1) = 0$

$\therefore y^2 - 1 = 0$

$\therefore y = \pm 1$

Case II: When y = 0

From (5), $x^2 + 0 - 2x = 0$

$\therefore x(x-2) = 0$

$\therefore x = 0$ or $x = 2$

$\therefore$ Stationary points are (1,1);(1,-1);(0,0);(2,0):

(i)At (1,1)

From (2), $r = 6(1) - 6 = 0$

$\therefore f$ is neither maximum or minimum at (1,1)

(ii)At (1,-1)

From (2), $r = 6(1) - 6 = 0$

$\therefore f$ is neither maximum or minimum at (1,-1)

(iii)At (0,0)

From (2), $r = 6(0) - 6 = -6 < 0$

From (3), $t = 6(0) - 6 = -6$

From (4), $s = 6(0) = 0$

$\therefore rt - s^2 = (-6)(-6) - 0 = 36 > 0$

$\therefore f$ has maximum at (0,0)

From (1), Maximum value of f

$\therefore f = (0)^3 + 3(0)(0)^2 - 3(0)^2 - 3(0)^2 + 4 = 4$

(iv)At (2,0)

From (2), $r = 6(2) - 6 = 6 < 0$

From (3), $t = 6(2) - 6 = 6$

From (4), $s = 6(0) = 0$

$\therefore rt - s^2 = (6)(6) - 0 = 36 > 0$

$\therefore f$ has maximum at (2,0)

From (1), Minimum value of f

$\therefore f = (2)^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 4 = 0$

Hence the function has

Maximum at (0,0) and Maximum value = 4

Minimum at (2,0) and Minimum value = 0

Q.3) **(20 marks)**

3a) Show that $\boldsymbol{x\,cosecx = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots}$ **.** **(6 marks)**

Ans: LHS = $x\,cosecx$

$$= \frac{x}{\sin x}$$

$$= \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

$$= \frac{x}{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)}$$

$$= \left[\left(1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\right)\right)\right]^{-1}$$

$$= 1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\right)^2 + \cdots \qquad \{\because (1-y)^{-1} = 1 + y + y^2 + y^3 + \cdots\}$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!}\right)^2 + \cdots$$

$$= 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$$

$$\boxed{\therefore x cosecx = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots}$$

3b) Reduce matrix to PAQ normal form and find 2 non-Singular matrices P & Q.

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

(6 marks)

Ans: $A_{3x4} = I_{3x3} \text{ x } A_{3x4} \text{ x } I_{4x4}$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1 \, ; R_3 - R_1;$ $$\rightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1 \, ; C_3 + C_1 \, ; C_4 - 2C_1 \, ;$ $$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 + C_2 \, ; \frac{1}{2} C_3$ $$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

LHS is the required PAQ form .

$$\boxed{\text{Here, } P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

3c) If $y=\cos(m\sin^{-1}x)$. Prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$. **(8 marks)**

Ans: $y = \cos(m\sin^{-1}x)$ ...(1)

Differentiating w.r.t. 'x', $y_1 = -\sin(m\sin^{-1}x)\cdot m\cdot\frac{1}{\sqrt{1-x^2}}$

$\therefore \sqrt{1-x^2}\cdot y_1 = -\ m\sin(m\sin^{-1}x)$

On Squaring, $(1-x^2)y_1^2=m^2\sin^2(m\sin^{-1}x)$

$\therefore (1-x^2)y_1^2=m^2\,[1-\ cos^2(msin^{-1}x)]$

$\therefore (1-x^2)y_1^2=m^2\,[1-\ y^2]$ (From 1)

Again differentiating w.r.t. 'x', $(1-x^2)2y_1y_2 + y_1^2\,(-2x) = m^2(0-2yy_1)$

$\therefore\ (1-x^2)y_2 - xy_1 = -m^2y$ (Dividing by $2y_1$)

Applying Leibnitz theorem, $\{y_n = u_nv + nu_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + {}^nC_3u_{n-3}v_3 + \cdots\}$

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}\right] - [xy_{n+1} + ny_n] = -m^2y_n$$

$$\therefore (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!} - xy_{n+1} - ny_n + m^2y_n = 0$$

$\therefore\ (1-x^2)y_{n+2} - xy_n(2n+1) + (-n^2 + n - n + m^2)y_n = 0$

$$\boxed{\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.}$$

Q.4) **(20 marks)**

4a) State and Prove Euler's Theorem for three variables. **(6 marks)**

Ans: Euler's theorem:

Statement: If 'u' is a homogenous function of three variables x, y, z of degree 'n' then Euler's theorem

States that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Proof:

Let $u = f(x, y, z)$ be the homogenous function of degree 'n'.

Let X = xt, Y = yt, Z = zt

$\therefore \frac{\partial X}{\partial t} = x;\ \frac{\partial Y}{\partial t} = y;\ \frac{\partial Z}{\partial t} = z$ ...(1)

At t = 1, ...(2)

X = x, Y = y, Z = z

$\therefore \frac{\partial f}{\partial X} = \frac{\partial f}{\partial x},;\ \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial y},;\ \frac{\partial f}{\partial Z} = \frac{\partial f}{\partial z},;$ ...(3)

Now, $f(X, Y, Z) = t^n f(x, y, z)$ ...(4)

$\therefore f$ → X, Y, Z → x, y, z, t

Differentiating (4) partially w.r.t. ‘t’, $\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial t} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial t} = nt^{n-1}f(x,y,z)$

$\therefore \frac{\partial f}{\partial X}\cdot x + \frac{\partial f}{\partial Y}\cdot y + \frac{\partial f}{\partial Z}\cdot z = n(1)^{n-1}f(x,y,z)$ (From 1,2 & 3)

$$\boxed{\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu}$$

4b) Show that all roots of $(x + 1)^6 + (x - 1)^6 = 0$ are given by -icot$\frac{(2k+1)n}{12}$ where k=0,1,2,3,4,5.

(6 marks)

Ans: $(x + 1)^6 + (x - 1)^6 = 0$

$\therefore (x + 1)^6 = -(x - 1)^6$

$\therefore \frac{(x+1)^6}{(x-1)^6} = -1$

$\therefore \left(\frac{x+1}{x-1}\right)^6 = e^{i\pi}$ $\{\because e^{i\pi} = cos\pi + isin\pi = -1 + i(0) = -1\}$ (Principal value)

$\therefore \left(\frac{x+1}{x-1}\right)^6 = e^{i(\pi+2k\pi)}$, k= 0, 1, 2, 3, 4, 5 (General Value)

$\therefore \frac{x+1}{x-1} = e^{i\pi(1+2k)/6}$ ...(1)

Let $2\theta = \frac{\pi(1+2k)}{6}$ ...(2)

$\therefore$ From (1) & (2), $\frac{x+1}{x-1} = e^{i2\theta}$

$\therefore$ By Componendo – Dividendo, $\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i2\theta}+1}{e^{i2\theta}-1}$

$$\therefore \quad \frac{2x}{2} = \frac{e^{i\theta}[e^{i\theta}+e^{-i\theta}]}{e^{i\theta}[e^{i\theta}-e^{-i\theta}]}$$

$\therefore \quad x = \frac{2cos\theta}{2isin\theta}$ $\qquad \left\{\because sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \; and \; cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\right\}$

$\therefore$ x = $\frac{1}{i}$ cotθ

$\boxed{\therefore x = -icot\frac{(2k+1)n}{12}}$ (From 2) where k = 0, 1, 2, 3, 4, 5

4c) Show that the following equations: -2x + y + z = a, x - 2y + z = b, x + y - 2z = c have no solutions unless a +b + c = 0 in which case they have infinitely many solutions. Find these solutions when a=1, b=1, c=-2. (8 marks)

Ans: Part I:

-2x + y + z = a

x - 2y + z = b

x + y - 2z = c

Writing the equations in the matrix form,

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$R_3+(R_1+R_2)$ $\rightarrow$ $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b+c \end{bmatrix}$ …(1)

Augmented matrix [A|B] = $\left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$

Number of unknowns = n = 3

Rank of A (r_A) = Number of non-zero rows in A = 2

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when ‘a+b+c≠0’ upon which,

Rank of [A|B] = $(r_{AB}) = 3$

Case II: Infinite Solution

For which, $r_A = r_{AB} < n$ (i.e. < 3)

This is only possible, when ‘a+b+c=0’ upon which,

Rank of [A|B] = $r_{AB} = 2$

Part II: Put a= 1, b = 1, c = -2, in (1)

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$R_2 - R_1$; → $$\begin{bmatrix} -2 & 1 & 1 \\ 3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(2)}$$

Here, $n - r_A = 3 - 2 = 1$

We have to assume one unknown.

Let y = t (≠0)

On expanding (2), $3x - 3y = 0$

$\therefore x - y = 0$

$\therefore x = y = t$

And, $-2x + y + z = 1$

$\therefore -2t + t + z = 1$

$\therefore z = 1 + t$

Hence, the solution is

$\boxed{x = t, y = t, z = 1 + t \text{ (Infinite Solution)}}$

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Q5)

5a) If z = f(x. y). x = r cos θ, y = r sin θ. prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$ **(6 marks)**

Ans: x = r cos θ and y = r sin θ ...(1)

Differentiating partially w.r.t. 'θ'. $\frac{\partial x}{\partial \theta}$ = - rsinθ; $\frac{\partial y}{\partial \theta}$= rcosθ; ...(2)

Differentiating partially w.r.t.' r', $\frac{\partial x}{\partial r}$=cosθ; $\frac{\partial y}{\partial r}$=sinθ ...(3)

Now, z → x, y →r , θ

By Chain Rule, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial r}$

∴ $\frac{\partial z}{\partial r}$ = cosθ $\frac{\partial z}{\partial x}$ + sinθ $\frac{\partial z}{\partial y}$ (From 3) ...(4)

Similarly, By Chain Rule, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial \theta}$

∴ $\frac{\partial z}{\partial \theta}$ = - rsinθ $\frac{\partial z}{\partial x}$ + rcosθ $\frac{\partial z}{\partial y}$ (From 2) ...(5)

RHS = $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

= $\left(\cos\theta\ \frac{\partial z}{\partial x} + \sin\theta\ \frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\ \frac{\partial z}{\partial x} + r\cos\theta\ \frac{\partial z}{\partial y}\right)^2$ (From 4 & 5)

= $cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + 2sin\theta\frac{\partial z}{\partial x}\cdot cos\theta\frac{\partial z}{\partial y} + sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 + sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2 - 2sin\theta\frac{\partial z}{\partial x}\ cos\theta\frac{\partial z}{\partial y} + cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2$

= $\left(\frac{\partial z}{\partial x}\right)^2 (cos^2\theta + sin^2\theta)$+ $\left(\frac{\partial z}{\partial y}\right)^2 (cos^2\theta + sin^2\theta)$

= $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

= LHS

$$\boxed{\text{Hence, } \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}$$

5b) If coshx = secθ prove that (i) x = log(secθ+tanθ). (ii) θ=$\frac{\pi}{2}$tan^{-1}(e^{-x}) (6 marks)

Ans: (i) coshx = secθ

$$\therefore \frac{e^x+e^{-x}}{2} = \sec\theta \qquad \left\{\because \cosh x = \frac{e^x+e^{-x}}{2}\right\}$$

$$\therefore e^x + \frac{1}{e^x} = 2sec\theta$$

$$\therefore (e^x)^2 + 1 = 2sec\theta e^x$$

$$\therefore (e^x)^2 - 2sec\theta e^x + 1 = 0$$

$$\therefore e^x = \frac{2sec\theta \pm \sqrt{4sec^2\theta - 4}}{2} \qquad \left\{\because Using, x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right\}$$

$$\therefore e^x = \frac{2sec\theta \pm \sqrt{4(sec^2\theta - 1)}}{2}$$

$$\therefore e^x = \frac{2sec\theta \pm 2tan\theta}{2}$$

$$\therefore e^x = sec\theta \pm tan\theta$$

Considering only positive root,

$$\therefore e^x = sec\theta + tan\theta \qquad \text{...(1)}$$

$$\therefore \mathrm{x} = \log(\sec\theta + \tan\theta)$$

(ii) From (1), $e^x = \frac{1}{cos\theta} + \frac{sin\theta}{cos\theta}$

$$\therefore e^x = \frac{1+sin\theta}{cos\theta}$$

$$\therefore \frac{1}{e^x} = \frac{cos\theta}{1+sin\theta}$$

$$\therefore e^{-x} = \frac{sin\left(\frac{\pi}{2}-\theta\right)}{1+cos\left(\frac{\pi}{2}-\theta\right)}$$

Put α = $\frac{\pi}{2} - \theta$ ...(2)

$$\therefore e^{-x} = \frac{sin\alpha}{1+cos\alpha}$$

$$\therefore e^{-x} = \frac{2sin(\alpha/2)cos(\alpha/2)}{2cos^2(\alpha/2)} \quad \{\because 2sinAcosA = sin2A; 1 + cos2A = 2cos^2A\}$$

$$\therefore e^{x} = tan\left(\frac{\alpha}{2}\right)$$

$$\therefore tan^{-1}(e^{-1}) = \frac{\alpha}{2}$$

$$\therefore 2tan^{-1}(e^{-1}) = \alpha$$

$$\therefore 2tan^{-1}(e^{-1}) = \frac{\pi}{2} - \theta \quad \text{(From 2)}$$

$$\boxed{\therefore \theta = \frac{\pi}{2} - 2tan^{-1}(e^{-1})}$$

5c) Solve by Gauss Jacobi Iteration Method: 5x – y + z = 10, 2x + 4y = 12, x + y + 5z = -1.

(8 marks)

Ans: From 1st equation, 5x = 10 + y – z

$\therefore$ x = $\frac{1}{5}$ (10 + y – z) = 0.2(10 + y – z)

Similarly,

From 2nd equation, x + 2y = 6

$\therefore$ 2y = 6 – x

y = $\frac{1}{2}$ (6 – x) = 0.5(6 – x) and,

z = 0.2(- 1 – x – y) = -0.2(1 + x + y)

Iteration 1:

Put $x_0 = y_0 = z_0$

$\therefore x_1 = = 0.2(10 + y_0 - z_0) = = 0.2(10 + 0 - 0) = 2$

$\therefore y_1 = 0.5(6 - x_0) = 0.5(6 - 0) = 3$

$\therefore z_1 = -0.2(1 + x_0 + y_0) = -0.2(1 + 0 + 0) = -0.2$

Iteration 2:

Put $x_1 = 2; y_1 = 3; z_1 = -0.2$

$\therefore x_2 = 0.2(10 + y_1 - z_1) = 0.2(10 + 3 + 0.2) = 2.64$

$\therefore y_2 = 0.5(6 - x_1) = 0.5(6 - 2) = 2$

$\therefore z_2 = -0.2(1 + x_1 + y_1) = -0.2(1 + 2 + 3) = -1.2$

Iteration 3:

Put $x_2 = 2.64; y_2 = 2; z_2 = -1.2$

$\therefore x_3 = 0.2(10 + y_2 - z_2) = 0.2(10 + 2 = 1.2) = 2.64$

$\therefore y_3 = 0.5(6 - x_2) = 0.5(6 - 2.64) = 1.68$

$\therefore z_3 = -0.2(1 + x_2 + y_2) = -0.2(1 + 2.64 + 2) = -1.128$

Iteration 4:

Put $x_3 = 2.64; y_3 = 1.68; z_3 = -1.128$

$\therefore x_4 = 0.2(10 + y_3 - z_3) = 0.2(10 + 1.68 = 1.128) = 2.5615$

$\therefore y_4 = 0.5(6 - x_3) = 0.5(6 - 2.64) = 1.68$

$\therefore z_4 = -0.2(1 + x_3 + y_3) = -0.2(1 + 2.64 + 1.68) = -1.0640$

Iteration 5:

Put $x_4 = 2.5616; y_4 = 1.68; z_4 = -1.0640$

$\therefore x_5 = 0.2(10 + y_4 - z_4) = 0.2(10 + 1.68 + 1.0640) = 2.5488$

$\therefore y_5 = 0.5(6 - x_4) = 0.5(6 - 2.5616) = 1.7172$

$\therefore z_5 = -0.2(1 + x_4 + y_4) = -0.2(1 + 2.5616 + 1.68) = -1.0483$

Iteration 6:

Put $x_5 = 2.5488$; $y_5 = 1.7192$; $z_5 = -1.0483$

$\therefore x_6 = 0.2(10 + y_5 - z_5) = 0.2(10 + 1.7192 + 1.0483) = 2.5535$

$\therefore y_6 = 0.5(6 - x_5) = 0.5(6 - 2.5488) = 1.7256$

$\therefore z_6 = -0.2(1 + x_5 + y_5) = -0.2(1 + 2.5488 + 1.7192) = -1.0536$

Iteration 7:

Put $x_6 = 2.5535$; $y_6 = 1.7256$; $z_6 = -1.0536$

$\therefore x_7 = 0.2(10 + y_6 - z_6) = 0.2(10 + 1.7256 =1.0536) = 2.5558$

$\therefore y_7 = 0.57(6 - x_6) = 0.5(6 - 2.5535) = 1.7232$

$\therefore z_7 = -0.2(1 + x_6 + y_6) = -0.2(1 + 2.5535 + 1.7256) = -1.0558$

Iteration 8:

Put $x_7 = 2.5558$; $y_7 = 1.7232$; $z_7 = -1.0558$

$\therefore x_8 = 0.2(10 + y_7 - z_7) = 0.2(10 + 1.7232 =1.0558) = 2.5558$

$\therefore y_8 = 0.57(6 - x_7) = 0.5(6 - 2.5558) = 1.7221$

$\therefore z_7 = -0.2(1 + x_6 + y_6) = -0.2(1 + 2.5558 + 1.7232) = -1.0558$

Hence, by Gauss Jacobi Iteration Method, the solution is

$$\boxed{x = 2.5558,\ y = 1.7221,\ z = -1.0558}$$

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Q.6) **(20 marks)**

6a) Prove that cos^{-1}[tanh(log x)] = π – 2$\left(x - \frac{x^3}{3} + \frac{x^5}{5} \ldots\right)$ **(6 marks)**

Ans: We know, tanhθ=$\frac{sinh\theta}{cosh\theta}$

$$= \frac{\left(e^{\theta} - e^{-\theta}\right)/2}{\left(e^{\theta} + e^{-\theta}\right)/2}$$

$$\therefore \text{tanh}\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

Put θ=log x, …(1)

$$\therefore \text{tanh(logx)} = \frac{e^{\log x} - e^{-\log x}}{e^{logx} + e^{-\log x}}$$

$$= \frac{e^{\log x} - e^{\log x^{-1}}}{e^{logx} + e^{\log x^{-1}}}$$

$$= \frac{x - x^{-1}}{x + x^{-1}}$$

$$= \frac{x(1 - x^{-2})}{x(1 + x^{-2})} \quad \text{…(2)}$$

Let y = cos^{-1}[tanh(log x)]

$$= \cos^{-1}\left[\frac{1 - x^{-2}}{1 + x^{-2}}\right] \quad \text{(From 2)}$$

$$= \cos^{-1}\left[\frac{1 - \left(x^{-1}\right)^{-2}}{1 + \left(x^{-1}\right)^{-2}}\right]$$

Put x^{-1}=tanθ

$$\therefore y = \cos^{-1}\left(\frac{1 - tan^2\theta}{1 + tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}\left(\frac{1}{x}\right) \quad \text{(From 1)}$$

= $2\cot^{-1}x$

= $2\left(\frac{\pi}{2} - tan^{-1}x\right)$

= $\pi - 2\tan^{-1}x$

= $\pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots\right)$

Hence, $\boxed{\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots\right)}$

6b) If y = $e^{2x} sin\frac{x}{2} cos\frac{x}{2} sin3x$. Find y_n. **(6 marks)**

Ans: y = $e^{2x} sin\frac{x}{2} cos\frac{x}{2} sin3x$ x $\frac{2}{2}$

= $\frac{1}{2} e^{2x}\left[sin2\left(\frac{x}{2}\right)\right] sin3x$ $\{\because 2sinAcosA = sin2A\}$

= $\frac{1}{2} e^{2x} sinxsin3x$ x $\frac{2}{2}$

= $\frac{1}{4} e^{2x}[\cos(3x - x) - \cos(3x + x)]$ $\{\because 2sinAsinB = \cos(A - B) - \cos(A + B)\}$

$\therefore$ y = $\frac{1}{4}[e^{2x}cos2x - e^{2x}cos4x]$

Taking n^th^ order derivative, $y_n = \frac{1}{4}\left\{\frac{d^n}{dx^n}(e^{2x}cos2x) - \frac{d^n}{dx^n}(e^{2x}cos4x)\right\}$ ...(1)

We know, If y = $e^{ax}\cos(bx + c)$, y_n = $r^n e^{ax}\cos(bx + c + n\emptyset)$ ...(2)

Here a = 2, c = 0, b_1 = 2 and b_2 = 4

$\therefore$ $r_1 = \sqrt{a^2 + b_1^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 8^{1/2}$ and $r_2 = \sqrt{a^2 + b_2^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 20^{1/2}$ ...(3)

And, $\emptyset_1 = \tan^{-1}\frac{b_1}{a} = \tan^{-1}\frac{2}{2} = \tan^{-1}1 = \frac{\pi}{4}$ & $\emptyset_2 = \tan^{-1}\frac{b_2}{a} = \tan^{-1}\frac{2}{4} = \tan^{-1}\frac{1}{2}$ ...(4)

$\therefore$ From (1),(2),(3) and (4),

$Y_n = \frac{1}{4}\left\{\left(8^{1/2}\right)^n e^{2x}\cos(2x + 0 + n\emptyset_1) + \left(20^{1/2}\right)^n e^{2x}\cos(4x + 0 + n\emptyset_2)\right\}$

$\boxed{\therefore y = \frac{1}{4}e^{2x}\left[8^{n/2}cos\left(2x + \frac{n\pi}{4}\right) + 20^{n/2}\cos(4x + n\emptyset_2)\right]}$, where $\emptyset_2 = \tan^{-1}\frac{1}{2}$

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6c) (i) Evaluate $\lim_{x\to 0}(cot\ x)^{sin\ x}$. **(4 marks)**

(ii) Prove that log$\left[\frac{sin(x+iy)}{sin(x-iy)}\right]$ = 2itan^{-1}(cot x tanh y). **(4 marks)**

Ans: (i) Let L = $\lim_{x\to 0}(cot\ x)^{sin\ x}$

$\therefore$ log$L$ = log$\{\lim_{x\to 0}(cot\ x)^{sin\ x}\}$

= $\lim_{x\to 0}\{log(cot\ x)^{sin\ x}\}$

= $\lim_{x\to 0}\sin x\cdot\log(\cot x)$

= $\lim_{x\to 0}\frac{\log(\cot x)}{cosec\ x}$ $\qquad\left(\frac{\infty}{\infty}\right)$

= $\lim_{x\to 0}\frac{\frac{1}{\cot x}\cdot -cosec^2 x}{-cosec\ x\cot x}$ (L' Hospital's Rule)

= $\lim_{x\to 0}\tan x\cdot\frac{1}{\sin x}\cdot\tan x$

= $\lim_{x\to 0}\tan x\cdot\frac{1}{\sin x}\cdot\frac{\sin x}{\cos x}$

= tan 0 x $\frac{1}{\cos 0}$

$\therefore$ log L = 0

$\therefore$ L = e^0

$$\boxed{\therefore\lim_{x\to 0}(cot\ x)^{sin\ x} = 1}$$

(ii) Consider, log[sin(x+iy)]=log[sin x cos(iy) + cos x sin (iy)]

$\therefore$log[sin(x+iy)]=log[sin x cosh y + icos x sinh y] {$\because$cos(ix)=cosh x; sin (ix)=isinh x;}

$\therefore$ log[sin(x+iy)]=$\frac{1}{2}$log[sin^2 x cosh^2 y + cos^2 x sinh^2 y] + itan^{-1}$\left|\frac{\cos x\sinh y}{\sin x\cosh y}\right|$

$$\left\{\because\log(x+iy)=\frac{1}{2}log(x^2+y^2)+itan^{-1}\left|\frac{y}{x}\right|\right\}$$

$\therefore$ log[sin(x+iy)]=$\frac{1}{2}$log[sin^2 x cosh^2 y + cos^2 x sinh^2 y] + itan^{-1}|$\cot x\tanh y$|

Taking Conjugate, $\log[\sin(x - iy)] = \frac{1}{2}\log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] - i\tan^{-1}|\cot x \tanh y|$

Now, $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log[\sin(x + iy)] - \log[\sin(x - iy)]$

$$= \left\{\frac{1}{2}\log[\sin 2\ x \cosh 2\ y\ +\ \cos 2\ x \sinh 2\ y]\ +\ i\tan - 1|\cot x \tanh y|\right\} -$$

$$\left\{\frac{1}{2}\log[\sin 2\ x \cosh 2\ y\ +\ \cos 2\ x \sinh 2\ y] -\ i\tan - 1|\cot x \tanh y|\right\}$$

$= 2i\tan^{-1}(\cot x \tanh y)$

$$\boxed{\therefore \log\left[\frac{sin(x+iy)}{sin(x-iy)}\right] = 2i\tan^{-1}(\cot x \tanh y)}$$