APPLIED MATHEMATICS - IV SOLUTION

(CBCGS SEM -4 MAY 2018)

BRANCH – ELECTRONICS &TELECOMMUNICATION

Q1)

(a). Find the extremal of $\int_0^1 (xy + y^2 - 2y^2 - y') \, dx$. **[5]**

Solution:

we have $F = xy + y^2 - 2y^2y'$

$\therefore \frac{\partial F}{\partial y} = x + 2y - 4yy'$ and $\frac{\partial F}{\partial y'} = -2y^2$

Putting these values in the Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, we get

$$x + 2y - 4yy' - \frac{d}{dx}(-2y^2) = 0$$

$\therefore x + 2y - 4yy' + 4yy' = 0$ $\therefore x + 2y = 0$ $\therefore y = -x/2$

This is the required extremal.

(b.) Verify Cauchy – Schwartz inequality for the vectors. u = (-4,2,1) and v = (8,-4,-2) **[5]**

Solution:

we have $||u|| = \sqrt{16 + 4 + 1} = \sqrt{21}$ *and* $||v|| = \sqrt{64 + 16 + 4} = \sqrt{84}$

$\therefore ||u||\,||v|| = \sqrt{21}\sqrt{84} = 42$

And $|u.v| = | u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 |$

$= |(-4)(8) + (2)(-4) + (1)(-2)|$

$= |-32 -8 -2| = 42$

$\therefore \| u \| \ \| v \| = |u . v|$

By the Cauchy - Schwartz inequality we should have $| u . v | \leq \| u \| \| v \|$.

Hence Cauchy - Schwartz inequality holds good for the given vectors.

(c) If $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots\ldots \lambda_n$ are thr eigen values of A then show that $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \ldots\ldots\ldots \frac{1}{\lambda_n}$ are the eigen values of A^{-1} [5]

Solution:

If λ is an eigen value of A and X is the corresponding eigenvector then ,

$$AX = \lambda X \qquad \therefore X = A^{-1}(\lambda X) = \lambda (A^{-1} X)$$

$$\therefore \frac{1}{\lambda} X = A^{-1}X \qquad \therefore \frac{1}{\lambda} \text{ is an eigen value of } A^{-1}$$

Hence If $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots\ldots \lambda_n$ are the eigen values of A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots . \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.

(d) A random variable x has following probability mass distribution : [5]

x	**0**	**1**	**2**
P(X=x)	**$3c^3$**	**$4c - 10c^2$**	**$5c - 1$**

Solution :

Since $\sum P_i = 1$ we have $P(0) + P(1) + P(2) = 1$

$\therefore 3cx^3 + 4c - 10c^2 + 5c - 1 = 0$

$$c = \frac{1}{3}$$

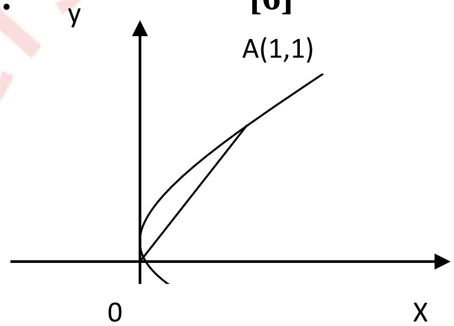
Probablity distribution is

X	0	1	2
P(X=x)	1/9	2/9	2/3

$P(x = 1) = P(x = 0) = 1/9$

Q2)

(a) Evaluate $\int_0^{1+i} z^2\,dz$ along (i) the line $y = x$ (ii) the parabola $x = y^2$ is the line integral independent of the path ? Explain . [6]



<u>Solution</u> :

Let OA be the line from z = 0 to z = 1 + i

(i) On the line OA i.e $y = x, dy = dx$

$\therefore dz = dx + idy = (1 + i)dx$

And x varies from 0 to 1

$\therefore \mathrm{I} = \int_0^{1+i}(\mathrm{x}+\mathrm{iy})^2\mathrm{dz} = \int_0^1 \llbracket(\mathrm{x}\rrbracket^2 - \mathrm{y}^2 + 2\mathrm{ixy})(1+\mathrm{i})\mathrm{dx}$

$= \int_0^1(\mathrm{x}^2 - \mathrm{x}^2 + 2\mathrm{ix}^2)(1+\mathrm{i})\mathrm{dx}\ [\because y = x]$

$= 2\mathrm{i}\,(1+\mathrm{i})\int_0^1 2\mathrm{i}\,(1+\mathrm{i})\left[\frac{\mathrm{x}^3}{3}\right]_0^1$

$= \frac{2}{3}\mathrm{i}\,(1+\mathrm{i}) = \frac{2}{3}(\mathrm{i} - 1)$

(ii) On the arc OA of the parabola $x = y^2, dx = 2ydy$

$$\therefore dz = dx + i\,dy = (2y + i)dy$$

$$I = \int_0^1 (x^2 - y^2 + 2ixy)(2y + i)dy$$

$= \int_0^1 (2\mathrm{y}^5 + 2\mathrm{y}^3 + 4\mathrm{i}\,\mathrm{y}^4 + \mathrm{iy}^4 - \mathrm{iy}^2 - 2\mathrm{y}^3)\mathrm{dy}$

$= \int_0^1 [(2\mathrm{y}^5 - 4\mathrm{y}^3) + \mathrm{i}\,(5\mathrm{y}^4 - \mathrm{y}^2)]\,\mathrm{dy}$

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$$= \left[\left(\frac{y^6}{3} - y^4 \right) + \ i \left(y^5 - \frac{y^3}{3} \right) \right]_0^1$$

$$= \ -\frac{2}{3} + \frac{2}{3} \text{i} \ \ = \frac{2}{3} (\text{i} - 1)$$

The two integrals are equal i.e the integral is independent of path because $f(z) = z^2$ is analytic function .

(b). A random variable x has following density function **[6]**

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} \boldsymbol{2e^{-2x}}, & \boldsymbol{x} > 0 \\ \boldsymbol{0}, & \boldsymbol{x} \leq \boldsymbol{0} \end{cases}$$ **find mgf , mean and variance .**

<u>Solution:</u>

We have

$$M_0(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \,.\, 2e^{-2x} \, dx$$

$$M_0(t) = \frac{2}{2-t} \ , \text{t} \neq 2$$

$$\text{Now, } M_o(t) = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1 - \frac{t}{2}\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \frac{t^3}{2^3} + \cdots$$

$\therefore \mu_1` =$ coefficient of $t = \frac{1}{2}$ ; $\mu_2` =$ coefficient of $\frac{t^2}{2!} = \frac{2}{2!}$

$\therefore$Mean $= \mu_1` = \frac{1}{2}$ ; var (X) $= \mu_2` - \mu_1`^2 = \frac{2}{2^2} - \frac{1}{2^2} = \frac{1}{4}$

(c). Calculate R (spearman's rank correlation) and r (karl pearson's) from the following data: **[8]**

x	**12**	**117**	**22**	**27**	**32**
y	**113**	**119**	**117**	**115**	**121**

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Solution : From the given data

$$\sum x = 110 , \sum y = 585 , N = 5 , \sum D^2 = 8$$

$$\overline{X} = 22 , \overline{Y} = 117 , \sum\left(X - \overline{X}\right)^2 = 250 ,\ \sum\left(Y - \overline{Y}\ \right)^2 = 40$$

$$\sum\left(X - \overline{X}\ \right)\left(Y - \ \overline{Y}\right) = 60$$

$$r = \frac{\sum\left(X - \overline{X}\ \right)\left(Y - \ \overline{Y}\right)}{\sqrt{\sum\left(X - \ \overline{X}\right)^2\ \sum\left(Y - \ \overline{Y}\right)^2}} \quad = \quad \frac{60}{\sqrt{250\ \times 40}} = 0.6$$

$$R \quad = \quad 1 - \ \frac{6\sum D^2}{N^3 - \ N} \quad = \quad 0.6$$

Here R = r

Thus value of R over r are equal. When arranged in ascending order also increases the same amount of 2 every time.

Q3)

a. **Let V= R^3 , show that W is a subspace of R^3 , where W = {(a,b,c):a + b +c = 0 } , that is W consists of all vectors where the sum of their components is zero.** **[6]**

Solution

0 = (0 ,0, 0) belongs to w , since 0+0+0 = 0

Suppose u = a, b, c and v = a', b', c' therefore any scalars k and k` we have

$$ku + kv \ = k(a, b, c) + \ k(a', b` , c`)$$

$$= (ka + ka`) + (kb + kb`) + (kc + kc`)$$

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Further more ,

$$(ka + k'a') + (kb + k'b') + (kc + kc') = k(a + b + c) + k'(a' + b' + c')$$
$$= 0$$

Thus $ku + kv$\` belong to W

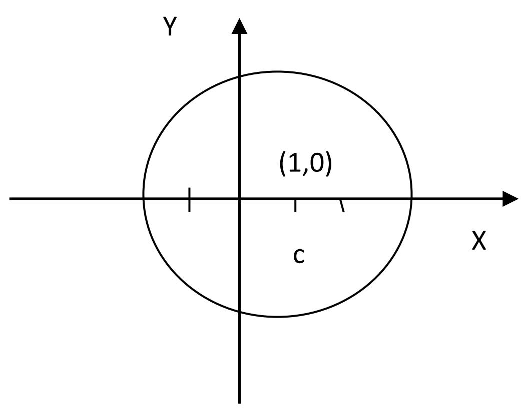
$\therefore$ W is the subspace of v

(b) . Evaluate $\int_c \frac{e^{2z}}{(z+1)^2}$ where c is the circle $|z - 1| = 3$. [6]

<u>Solution :</u>

The circle $|z - 1| = 3$ has centre at C (1,0) and radius 3 .

Further , z+1 = 0 gives A. $z = -1$. The point A lies inside the circle . Hence $e^{2z}/(z+1)^4$ is not analytic in C. We take $f(z) = e\text{^}2z$ which is analaytic in C..



By corollary of Cauchy's Formula

$$\int_c \frac{f(z)}{(z-z_0)} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f^3(z_0)$$

$$= \frac{2\pi i}{3!} \cdot \frac{8}{e^2} = \frac{8\pi i}{3e^2}$$

c) Show that matrix A is diagonizable . Also find the transforming matrix and the diagonal matrix where [8]

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

<u>Solution :</u>

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Let λ be the eigen value A.

The characteristic equation of $|A - \lambda I|$ = 0

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

On solving we get

λ^3(sum of diagonal elements) λ^2 + (sum of the minors of the diagonal elements) λ - $|A|$ = 0

∴ $\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$

∴ The eigen values (λ) are 3 , 3 , 5

Since , all the eigen values are distinct the matrix A is diagonalisable.

(i) For λ = 3 , [A – λI]X = 0 gives

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + 1x_2 - 1x_3 = 0$; $2x_1 + 2x_2 - 2x_3 = 0$

By cramer's rule : we get $x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Similarly when λ = 5 $x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Hence A is diagonalizable since it has 3 linearly independent eigen values.

Transforming Matrix = P = $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

Diagonal matrix , D = $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Q4)

(a) Find the extremal of $\int_{x_0}^{x_1}(2xy - y^{2}{}'')$ **[6]**

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Solution :

We have $f = 2xy - y''^2$

$\therefore \frac{\partial f}{\partial y} = 2x$, $\frac{\partial f}{\partial y'} = 0$, $\frac{\partial f}{\partial y} = -2y''$

Hence , the equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y\prime}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$ becomes

$$2x - 0 + \frac{d^2}{dx^2}(-2y'') = 0 \quad \therefore 2x - 2\frac{d^2y}{dx^2}\left(\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore \frac{d^4y}{dx^4} = x$$

This is a linear differential equation to the fourth order

Its A.E is $D^4 = 0$ $\therefore D = 0, 0, 0, 0$

$\therefore$ The CF is $y = c_1 + c_2x + c_3x^2 + c_4x^3$

And PI $y = \frac{1}{D^4}x$

$\therefore y = \frac{1}{D^2}\int x dx = \frac{1}{D^3}.\frac{x^2}{2} = \frac{1}{D^2}\int \frac{x^2}{2}\, dx$

$$= \frac{1}{D^2}.\frac{x^3}{3x2}dx = \frac{1}{D}\int \frac{x^3}{3.2}dx = \frac{1}{D}.\frac{x^4}{4.3.2}$$

$$= \int \frac{x^4}{4.3.2}dx = \frac{x^5}{5.4.3.2} = \frac{x^5}{5!}$$

Hence the solution is $y = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{x^5}{5!}$

(b) A transmission channel has a per – digit error probability p = 0.01 . Calculate probability of more than 1 error in 10 received digits using

(i) Binomial Distribution (ii) Poission Distribution **[6]**

Solution:

(i) **Binomial Distribution :** We have p = 0.01 , q = 1 − p = 0.99 , n= 10

$$\therefore P(X - x) = {}^{n}C_{x}p^{x}q^{n-x} = 10^{10}C_{x}(0.01) \times (0.99)^{10-x}$$

$$P(X > 1) = 1 - P(X \le 1) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - {}^{10}C_{0}(0.01)^{0}(0.99)^{10} - {}^{10}C_{1}(0.01)^{1}(0.99)^{9}$$

$$= 1 - 0.9044 - 0.09135 = 0.00425$$

(ii) **Poisson Distribution :** We have $m = np$= 10(0.01) = 0.1

$$\therefore P(X = x) = e^{-m}.\frac{m^{x}}{x!} = e^{-0.1}\frac{(0.1)^{x}}{x!}$$

$$\therefore P\ (X > 1) = 1 - P(X \le 1)$$

$$= 1 - P(X = 0) - P(X = 1)$$

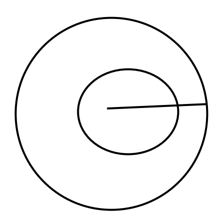
$$= 1 - e^{-0.1}\frac{(0.1)^{0}}{0!} - e^{-0.1}\frac{(0.1)^{0}}{1!}$$

$$= 1 - 0.9048 - 0.0905 = 0.0047$$

(c) **Obtain the Taylor's and Laurent's expansion of $f(z) = \frac{z-1}{z^{2}+2z-3}$ indicating regions of convergence**. **[8]**

Solution :

Let $f(z) = \frac{z-1}{(z+1)(z-3)} = \frac{y_2}{z+1} + \frac{y_2}{z-3}$



f(z) is not analaytic at $z = -1$. and $z = 3$

f(z) is analytic when |z| < 1 , 1 < |z < 1 , |z| > 3

|0 < |z| < 1

Case 1: When $|z| < 1$ & $|z| < 3$

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$$f(z) = \frac{1}{1+z} + \frac{1}{2}\frac{1}{(-3)} \cdot \frac{1}{1-\left(\frac{z}{3}\right)}$$

$$= \frac{1}{2}\,(1+z)^{-1} - \frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{3} - \frac{5}{9} + \frac{13}{27}z^2 + \cdots ..$$

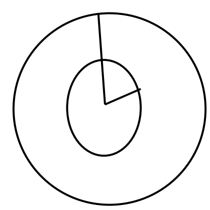
<u>Case (ii)</u> : When $1 < |z| < 3$ we get $\left|\frac{1}{z}\right| < 1$ & $\left|\frac{z}{3}\right| < 1$

$$f(z) = \frac{1}{2} \cdot \frac{1}{1+z} + \frac{1}{2} \cdot \frac{1}{z-3}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right] - \frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots\right]$$

This is the required Laurent's series.



<u>Case (iii):</u> When $|z| > 3$, *clearly* $|z| > 1$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-1} + \frac{1}{2}\,\frac{1}{z-3}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} + \frac{1}{2z}\left(1-\frac{3}{2}\right)^{-1}$$

$$= \frac{1}{2z}\left[2 + \frac{2}{z} + \frac{10}{z^2} + \frac{26}{z^3} + \cdots\right]$$

This is the required laurentz series

Q5)

(a)

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Show that matrix A = $\begin{bmatrix} \mathbf{0} & \boldsymbol{c} & \boldsymbol{-b} \\ \boldsymbol{-c} & \mathbf{0} & \boldsymbol{a} \\ \boldsymbol{b} & \boldsymbol{-a} & \mathbf{0} \end{bmatrix}$ **satisfies Cayley – Hamilton Theorem and hence find A^{-1} if it exists.** **[6]**

Solution:

The characteristic Equation is $\begin{bmatrix} 0-\lambda & c & -b \\ -c & 0 & a \\ b & -a & 0-\lambda \end{bmatrix} = 0$

$\therefore -\lambda\,(\lambda^2 + a^2) - \ c(c\lambda - ab) - \ b(ac + b\lambda) = \ 0$

$\therefore -\lambda^3 - \ \lambda a^2 - c^2\lambda + abc - abc - b^2\lambda = 0$

$\therefore\ \lambda^3 + \ (a^2 + \ b^2 + c^2)\lambda = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(1)$

Now ;

$$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}$$

$$\therefore A^3 = \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & bc & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3 - cb^2 - ca^2 & b^3 + bc^2 + ba^2 \\ c^3 + \ ca^2 + cb^2 & 0 & -ab^2 - ac^2 - a^3 \\ -bc^2 - \ b^3 - a^2b & ac^2 + ab^2 + a^3 & 0 \end{bmatrix}$$

$$A^3 = -\,(a^2 + b^2 + c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \ -(a^2 + b^2 + c^2)A$$

$A^3 + (a^2 + b^2 + c^2)A = 0 \quad \therefore A\ satisfies\ the\ equation\ (1)$

Hence ,A satisfies Cayley – Helminton Theorem

Now the determinant of the matrix A

i.e $|A| = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & bc & 0 \end{vmatrix} = 0\,(0 + a^2) - c(0 - ab) - b(ac - 0)$

$= abc - abc = 0$ Since the matrix A is singular A^{-1} does not exist.

(b) **Let R^3 have the Euclidean inner product . Use Gram –Schamidit process transform the basics $\{u_1, u_2, u_3\}$ into an orthonomal basis where $u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$ [6]**

Solution:

Step 1: $v_1 = u_1 = (1,1,1)$

Step 2 : $v_2 = u_2 - \text{proj } u_2 = u_2 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} \cdot v_1$

Now, $\langle u_2, v_1\rangle = -1 + 1 = 0;$ and $\|v_1\|^2 = 1 + 1 + 1 = 3$

$\therefore v_2 = u_3 - proj\; u_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3, v_2\rangle}{\|v_1\|^2} \cdot v_2$

$$\therefore v_3 = (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0) = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

Hence , $v_1 = (1,1,1), v_2 = (-1,1,0), v_3 = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$ from the orthogonal basis of R^3.

Now the norm of these vectors are $\|v_1\| = \sqrt{1+1+1} = \sqrt{3}$,

$\|v_2\| = \sqrt{1+1+0} = \sqrt{2}$; $\|v_3\| = \sqrt{\frac{1}{36} + \frac{1}{36} + \frac{1}{9}} = \sqrt{\frac{6}{36}} = \frac{1}{\sqrt{6}}$

Hence the orthonomal basis of R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \quad q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right);$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) \quad \left[\because \frac{\sqrt{6}}{3} = \frac{\sqrt{6}\sqrt{6}}{3\sqrt{6}} = \frac{\sqrt{6}}{3\sqrt{6}} = \frac{2}{\sqrt{6}}\right]$$

(c) The marks obatained by 1000 students in an examination are found to be normally distributed with mean 70 and the standered deviation 5. Estimate the number of students whose marks will be [8]
i. between 60-75 ii. More than 75.

Solution : We have S.N.V. $Z = \frac{X-m}{\sigma} = \frac{X-70}{5}$

(i) When $X = 60 , Z = \frac{60-70}{5} = -2$;

When $X = 75 , Z = \frac{75-70}{5} = 1$;

$$P\,(60 \le X \le 75) = P(-2 < Z < 1)$$

$= \text{area between } (z = -2\ \&\ z = 1)$

$= \text{Area from } (z = 0 \text{ to } z = 2) + \text{area from } (z = 0\ to\ z = 1)$

$= 0.4772 + 03413 = 0.8185$

$\therefore$ Number of students getting marks between 60 and 75

$= NP = 1000 \times 0.8185 = 818$

(ii) $P(X \ge 75) = P(Z \ge 1)$
$= \text{Area to the right of } Z = 1$
$= 0.5 - (area\ between\ Z = 0\ and\ Z = 1)$
$= 0.5 - 0.3413 = 0.1587$

$\therefore$ Number of students getting more than 75 marks
$= Np = 1000 \times 0.1587 = 159$

Q6)

(a) Using the Rayleigh – Ritz method , solve the boundary value problem
$\boldsymbol{I = \int_0^1 \left(2xy + y^2 - y'^2\right)dx \,;\, 0 \le x \le 1 \,,\, given\ \ y(0) = y(1) = 0}$ **[6]**

Solution:

We have to extremise $I = \int_0^1 F(x, y, y')dx$ ....(1)

$where\ \ F = 2xy + y^2 - y'^2$.....................................(2)

Now assume the trial solution $\bar{y}(x) = c_0 + c_1x + c_2x^2 \ldots\ldots$ (3)

By the data $\bar{y}(0) = 0 \therefore c_0 = 0$; $\bar{y}(1) = 0 \quad \therefore 0 = c_1 + c_2 \ \therefore c_2 - c_1$

$\therefore \bar{y}(x) = c_1x - c_2x^2 = c_1x(1 - x)$.............................(4)

$\bar{y}\,'(x) = c_1x - 2c_1x = c_1(1 - 2x)$

Putting these values in $I = \int_0^1 (2xy + y^2 - y'^2)dx$ we get

$$I = \int_0^1 \{2x[c_1x(1 - x)] + c_1^2x^2(1 - x)^2 - c_1^2(1 - 2x)^2\}dx$$

$$= c_1 \int_0^1 \{2(x^2 - x^3) + c_1[x^2 - 2x^3 + x^4 - (1 - 4x + 4x^2)]\}dx$$

$$= c_1 \int_0^1 \{2(x^2 - x^3) + c_1[-1 + 4x - 3x^2 - 2x^3 + x^4]\}dx$$

$$= c_1 \left[2\left(\frac{x^3}{3} - \frac{x^4}{4}\right) + c_1\left(-x + 2x^2 - x^3 - \frac{x^4}{2} + \frac{x^5}{5}\right)\right]_0^1$$

$$= c_1 \left[2\left(\frac{1}{3} - \frac{1}{4}\right) + c_1\left(-1 + 2 - 1 - \frac{1}{2} + \frac{1}{5}\right)\right]$$

$\therefore I = c_1\left(\frac{1}{6} - \frac{3}{10}c_1\right) = \frac{c_1}{6} - \frac{3}{10}c_1^2$

Its stationary values are given by :

$$\frac{dI}{dc_1} = 0 \quad \therefore \frac{1}{6} - \frac{3}{5}c_1 = 0 \quad \therefore c_1 = \frac{1}{6}\cdot\frac{5}{3} = \frac{5}{18}.$$

Hence from (4) the approximate solution is $\bar{y}\,(x) = \frac{5}{18}\,x(1 - x)$

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(b) Show that A= $\begin{bmatrix} \mathbf{4} & \mathbf{-2} & \mathbf{2} \\ \mathbf{6} & \mathbf{-3} & \mathbf{4} \\ \mathbf{3} & \mathbf{-2} & \mathbf{3} \end{bmatrix}$ **is derogatory and find its minimal polynomial.** **[6]**

Solution:

The characteristic equation of A is

$$\begin{bmatrix} 4-\lambda & -2 & 2 \\ 6 & -3-\lambda & 4 \\ 3 & -2 & 3-\lambda \end{bmatrix} = 0$$

Further solving the equation we get

$$\therefore\ 4-\lambda[(-3-\lambda)(3-\lambda)+8\,]-2\,[6(3-\lambda)-12]+2\,[(-12)+9+3\lambda]$$

$$\therefore (\lambda-1)(\lambda^2-3\lambda+2) \quad \therefore \lambda = 1{,}1.2$$

Now finding the minimal polynomial of A . As we know that each characteristics is root of A is also a root of the minimal polynomial of A. So if f(x) is the minimal polynomial of A , then $x-2\ and\ x-1\ are\ factors\ of\ f(x)$

Now checking whether $(x-2)(x-1) = x^2-3x+8$ annhilates A

Now , $A^2 - 3A + 2i = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} - 3\begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 10 & -6 & 6 \\ 18 & -11 & 12 \\ 9 & -6 & 7 \end{bmatrix} - 3\begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore f(x) = x^2 - 3x + 8$ annihilates A . Thus ,$f(x)$ is the monic polynomial of lowest degree that annihilates A.

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Hence ,$f(x)$ is the minimal polynomial of A. Since the degree of f(x) is less than the order of A. A is derogatory

c) (i) **[4]**

Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

Solution:

Let $e^{i\theta} = z \therefore e^{i\theta} \cdot i\,d\theta = \; d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z^2-1}{2iz}$

$$I = \int_c \frac{1}{5+3\left(\frac{z^2-1}{2iz}\right)} \cdot \frac{dz}{iz} = \int_c \frac{2}{3z^2 + 10iz - 3} dz$$

$= \int_c \frac{2}{(3z+i)(z+3i)}\, dz$. where C is the centre $|z| = 1$

Now the poles of f(z) are given by (3z +i)(z+3i) = 0

$\therefore z = -\frac{i}{3}$ and $z = -3i$ are simple poles. But $z = -\frac{i}{3}$ lies inside and $z = -3i$ are simple poles . But $z = -\frac{i}{3}$ lies inside and $z = -3i$ lies outside the circle $|z| = 1$.

Residue $\left(at \;\; z = -\frac{i}{3}\right) = \lim_{z\to -\frac{i}{3}} \left[z + \left(\frac{i}{3}\right)\right] \cdot \frac{2}{(3z+i)(z+3i)}$

$$= \lim_{z\to -\frac{i}{3}} \frac{2}{3(z+3i)} = \frac{1}{4i}$$

$$\therefore I = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

(b) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$, $a > 0, b > 0$ **[4]**

Solution:

(i) Consider as before the contour consisting of a semi-circle and diameter on the real axis with the centre of the origin.

(ii) $\rightarrow$ Now , $zf(z) = \frac{z^3}{[(z)]^2+a^2)(z^2+b^2)}$ $0 \; as \; |z| \rightarrow \infty$

(iii) Now , $(z^2+a^2)(z^2+ b^2) = 0 \; i.e \; z = +ai, \pm ai, +bi, -bi$

One of these $z = ai$, $z = bi$ lie in the upper of the z plane

(iv) Residue (at z = ai) $= \lim_{z \to ai} (z - a\,i) \cdot \left(\frac{z^2}{(z-ai)(z+ai)(z^2+b^2)}\right)$

$$= \frac{-a^2}{2ai(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Similarly Residue (at z = bi) $= \frac{-b^2}{2bi(a^2-b^2)} = \frac{-b}{2i(a^2-b^2)}$

(v) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = 2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)}\right] = \frac{\pi}{a+b}$

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