COMPUTER ENGINEERING

APPLIED MATHEMATICS – 3

(CBCGS – DEC 2018)

Q1] a) If Laplace Transform of $erf(\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$ then find $L\{e^t.erf(2\sqrt{t})\}$ (5)

Solution:-

Given :- $L[erf(\sqrt{t})] = \frac{1}{s\sqrt{s+1}}$

$L[erf(2\sqrt{t})] = L[erf(\sqrt{4t})]$

By change of scale property ; $\left\{L[f(at)]= \frac{1}{a}\varphi(\frac{s}{a})\right\}$

$$L[erf(2\sqrt{t})] = \frac{1}{4} \times \frac{1}{\left(\frac{s}{4}\right)\sqrt{\left(\frac{s}{4}\right)+1}}$$

$$= \frac{2}{s\sqrt{s+4}} = \varphi(-s) \quad \text{................ (1)}$$

$L[e^f. erf(2\sqrt{t})]$ can be found by first shifting theorem,

$\{L[e^{at}f(t)]= \varphi(s-a)\}$

$L[e^t erf(2\sqrt{t})] = \varphi(s-1)$

From I; replace s by s-1

We get $L[e^t erf(2\sqrt{t})] = \frac{2}{(s-1)\sqrt{(s-1)+4}} = \frac{2}{(s-1)\sqrt{s+3}}$

$$\mathbf{L[e^t erf(2\sqrt{t})] = \frac{2}{(s-1)\sqrt{s+3}}}$$

Q1] b) Find the orthogonal trajectory of the family of curves given by $e^{-x}\cos y + xy = C$ (5)

Solution:-

Let $u = e^{-x}\cos y + xy$;

To find orthogonal trajectory of u = C

i.e. find v(hormonal conjugate of u)

$u_x = -e^{-x}\cos y + y$ ……………… [differentiating partially wrt x]

$u_y = -e^{-x}\sin y + x$ ……………….. [differentiating partially wrt y]

$f'(z) = u_x + iv_x = u_x - iu_y$ ……………[by CR eqn ; $v_x = -u_y$

By Milne-Thompson's method ; replace x = z; y = 0

$f'(z) = -e^{-z}\cos(0) + (0) - i[-e^{-z}\sin(0) + z] = -e^{-z} - iz$

By integrating both sides;

$$f(z) = \frac{-e^{-z}}{-1} - \frac{iz^2}{2} + c = e^{-z} - \frac{iz^2}{2} + c$$

put z = x + iy

$$f(z) = e^{-(x+iy)} - \frac{i(x+iy)^2}{2} + C$$

$$f(z) = e^{-x}.e^{-iy} - \frac{i}{2}[x^2-y^2+2xiy] + C$$

$$f(z) = e^{-x}(\cos y - i\sin y) - \frac{i}{2}[x^2-y^2+2xiy] + C$$

Imaginary part ; $v = -e^{-x}\sin y - \frac{1}{2}[x^2-y^2]$

Hence required orthogonal trajectory = $-e^{-x}\sin y - \frac{1}{2}[x^2-y^2]$

Q1] c) Find Complex form of Fourier Series for e^{2x}; 0 < x < 2 **(5)**

Solution:-

In interval (0,2l); $f(x) = e^{2x}$

$F(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$ where $C_n = \frac{1}{2l}\int_0^{2l} f(x)e^{in\pi x/l}dx$

Put l = 1 therefore in interval (0 < x < 2)

We get $f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x}$; $C_n = \frac{1}{2}\int_0^2 f(x)e^{in\pi x}dx$

$$C_n = \frac{1}{2}\int_0^2 e^{2x}.e^{in\pi x}dx = \frac{1}{2}\int_0^2 e^{(2-in\pi)x}dx$$

$$C_n = \frac{1}{2}\left[\frac{e^{(2-in\pi)x}}{(2-in\pi)x}\right]_0^2 = \frac{1}{2}\left[\left[\frac{e^{(2-in\pi)x}}{(2-in\pi)x}\right] - \frac{1}{(2-in\pi)}\right] = \frac{1}{2}\left[\frac{(e^4 \times e^{i2n\pi})-1}{2-in\pi}\right] = \frac{1}{2}\left[\frac{e^4-1}{2-in\pi}\right]$$

$$C_n = \frac{e^4-1}{4-2in\pi}$$

$$e^{2x} = \sum_{-\infty}^{\infty} C_n e^{in\pi x} = \sum_{-\infty}^{\infty}\left[\frac{e^4-1}{4-2in\pi}\right].e^{in\pi x}$$

$$e^{2x} = (e^4-1)\sum_{-\infty}^{\infty}\frac{e^{in\pi x}}{4-2in\pi}$$

Q1] d) If the regression equations are x - 6y + 90 = 0 ; 15x - 8y - 180 = 0. Find the means of x and y, correlation coefficients and standard derivation of x if variance of y = 1 **(5)**

Solution:-

Given equation:- 5x - 6y + 90 = 0 ; 15x - 8y - 180 = 0

(1) Means of x,y:

Solving the equation n simultaneously,

5x - 6y = -90

$15x - 8y = 180$

We get, $X = 36$; $Y = 45$

(2) Correlation coefficients

Suppose the first equation represents the lines of regression of X on Y

Writing it as $X = \frac{6Y}{5} - \frac{90}{5} = b_{xy} = \frac{6}{5}$

Suppose the second equation represents the lines of regression of Y on X

Writing it as $Y = \frac{15X}{8} - \frac{180}{8} = b_{yx} = \frac{15}{8}$

$$r = \sqrt{b_{yx}.\, b_{xy}} = \sqrt{\frac{6}{5} \times \frac{15}{8}} = 1.5$$

But r cannot be greater than 1.

Hence our assumption is wrong;

Treating equation 1 as line of regression of Y on X and equation 2 as line of regression of X on Y .

$Y = \frac{5X}{6} + \frac{90}{6} = b_{yx} = \frac{5}{6}$

$X = \frac{8Y}{15} + \frac{180}{15} = b_{xy} = \frac{8}{15}$

$$r = \sqrt{b_{yx}.\, b_{xy}} = \sqrt{\frac{5}{6} \times \frac{8}{15}} = 0.6667$$

(3) To find σ_x; given $\sigma_y^2 = 1$

$\sigma_y^2 = 1$

$\sigma_y = 1$

$b_{yx} = r \times \frac{\sigma_y}{\sigma_x}$

$$\frac{5}{6} = \frac{2}{3}\times\frac{\sigma_y}{\sigma_x}$$

$$\frac{15}{12} = \frac{1}{\sigma_x}$$

$$\sigma_x = \frac{12}{15} = 0.8$$

$\bar{X} = 36$; $\bar{Y} = 45$

r = 0.6667

$\sigma_x = 0.8$

Q2] a) Show that the function is Harmonic and find the Harmonic conjugate $v = e^x \cos y + x^3 - 3xy^2$ **(6)**

Solution:-

Given:-

$$\frac{\partial v}{\partial x} = e^x \cos y + 3x^2 - 3y^2$$

$$\frac{\partial^2 v}{\partial x^2} = e^x \cos y + 6x \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial v}{\partial y} = -e^x \sin y - 6yx$$

$$\frac{\partial^2 v}{\partial y^2} = -e^x \cos y - 6x \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x \cos y + 6x - e^x \cos y - 6x = 0$$

Therefore, v satisfies Laplace's equation

v is harmonic

Finding harmonic conjugate , u;

$V_y = \psi_1(x,y)$ and $V_x = \psi_2(x,y)$

$\psi_1(z,0) = -e^z.0 - 6(z).0 = 0$

$\psi_2(z,0) = e^z + 3z^2$

$f'(z) = \psi_1(z,0) + i\psi_2(z,0) = i(e^z + 3z^2)$

On integrating; $f(z) = i[e^z + z^3] = i[e^{(x+iy)} + (x+iy)^3] = i[e^x.e^{iy} + (x+iy)^3]$

$f(z) = i[e^x\{\cos y + i\sin y\} + x^3 + 3x^2iy - 3xy^2 - iy^3]$

Real part; $u = -e^x\sin y - 3x^2y + y^3$

Harmonic conjugate = $-e^x\sin y - 3x^2y + y^3$

Q2] b) Find Laplace Transform of:-

$f(t) = \begin{cases} t, & 0<t<1, \\ 0, & 1<t<2 \end{cases}$ $\quad f(t+2) = f(t)$ **(6)**

Solution:-

f(t) is periodic with period a = 2; we have

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt$$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[\int_2^0 e^{-st}tdt + \int_1^2 0dt\right] = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}tdt\right]$$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[t\left(\frac{-e^{-st}}{s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^1 = \frac{1}{1-e^{-2s}}\left[\left(\frac{-e^{-s}}{s}\right) - \left(\frac{-e^{-s}}{s^2}\right) + \frac{e^{-s}}{s^2}\right]$$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[\left(\frac{-e^{-s}}{s}\right) - \frac{-e^{-s}+1}{s^2}\right]$$

$$L[f(t)] = \frac{1}{s^2(1-e^{-2s})}[-se^{-s}+1-e^{-s}]$$

$$\mathbf{L[f(t)] = \frac{1}{s^2(1-e^{-2s})}[1-e^{-s}-se^{-s}]}$$

Q2] c) Find Fourier expansion of $f(x) = -x^2$; $-1<x<1$ (8)

Solution:-

$f(x) = x - x^2 \quad ; -1<x<1$

Given function is difference b/w odd and even function

$f(x) = f_1(x) - f_2(x)$

Here , $l = 1$

For $f_1(x) = x$ which is odd ; $a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad = \quad 2\int_0^1 x\sin(n\pi x)dx$$

$$b_n = 2\left[x\left\{\frac{-\cos n\pi x}{n\pi}\right\}-(1)\left\{\frac{-\sin(n\pi x)}{n^2\pi^2}\right\}\right]_0^1 = 2\left[1\left\{\frac{-\cos n\pi}{n\pi}\right\}-1\left\{\frac{-\sin(n\pi)}{n^2\pi^2}\right\}-\left\{\frac{\sin 0}{n^2\pi^2}\right\}\right]$$

$$b_n = 2\left[\frac{-(1)^n}{n\pi}\right]$$

For $f_2(x) = x^2$ which is even ; $b_n = 0$

$$a_0 = \frac{1}{l}\int_0^l f(x)dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \quad \text{................ (2)}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = 2\int_0^1 x^2\cos(n\pi x)dx$$

$$a_n = 2\left[x^2\left(\frac{\sin(n\pi x)}{n\pi}\right)-2x\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)+2\left(\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\right]_0^1$$

Solving equation we get,

$$a_n = \frac{4(-1)^n}{n^2\pi^2} \quad \text{....................... (3)}$$

$f(x) = f_1(x) - f_2(x)$

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right) - \left\{a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)\right\}$$

$$\mathbf{f(x) = \frac{-2}{\pi}\sum\frac{(-1)^n}{n}\sin(n\pi x) - \frac{1}{3} + \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos(n\pi x)}$$

Q3] a) Find the analytic function f(z) = u + iv if $v = \log(x^2+y^2) + x - 2y$ (6)

Solution:-

$v = \log(x^2+y^2) + x - 2y$

Differentiating partially with respect to x and y

$$\frac{\partial v}{\partial x} = \frac{1}{(x^2+y^2)} \times 2x + 1 \quad \ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial v}{\partial y} = \frac{1}{(x^2+y^2)} \times 2y - 2 \quad \ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial v}{\partial y} = \psi_1(x,y) \quad \text{and} \quad \frac{\partial v}{\partial x} = \psi_2(x,y)$$

$$\psi_1(z,0) = \frac{0}{(Z^2)} - 2 \quad ; \quad \psi_2(z,0) = \frac{2z}{(Z^2)} + 1$$

$$f'(z) = \psi_1(z,0) + i\,\psi_2(z,0)$$

Integrating both sides;

$$f(z) = -\int 2dz + i\int \frac{2z}{Z^2} + 1dz = -2z + i\int \frac{2}{z} + 1dz$$

$$f(z) = -2z + i(2\log z + z) = -2z + 2i\log z + iz$$

$$\mathbf{f(z) = z(i-2) + 2i\log z}$$

Q3] b) Find inverse z transform of

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} \quad ; \quad 3 < |z| < 4 \qquad (6)$$

Solution:-

By partial fraction:-

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-4)}$$

$$3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

Put z = 4

$3(4)^2 + 26 - 18(4) = 0(A) + 0(B) + C(4-2)(4-3)$

$2 = C(2)(1)$

$C = 1$

Put Z = 3

$3(2)^2 - 18(2) + 26 = A(2-3)(2-4) + 0(B) + 0(C)$

$2 = A(-1)(-2)$

$A = 1$

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{1}{(z-2)} + \frac{1}{(z-3)} + \frac{1}{(z-4)}$$

Since |z| >3 we take common z from first two terms and 4>|z| we take 4 common from last term.

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{1}{z(1-\frac{2}{z})} + \frac{1}{z(1-\frac{3}{z})} + \frac{1}{4(\frac{z}{4}-1)}$$

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{1}{z(1-\frac{2}{z})} + \frac{1}{z(1-\frac{3}{z})} - \frac{1}{4(1-\frac{z}{4})}$$

RHS:-

$$= \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2 + ---\right] + \frac{1}{z}\left[1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2 + ---\right] - \frac{1}{4}\left[1+\left(\frac{z}{4}\right)+\left(\frac{z}{4}\right)^2 + --\right]$$

$$= \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2 + ---\left(\frac{2}{z}\right)^{k-1} + ---\right] + \frac{1}{z}\left[1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2 + ---\left(\frac{3}{z}\right)^{k-1} +\right] - \frac{1}{4}\left[1+\left(\frac{z}{4}\right)+\left(\frac{z}{4}\right)^2 + ---\left(\frac{z}{4}\right)^{k-1} + --\right]$$

Coefficient of z^{-k} in 1st term = 2^{k-1} ; $k \geq 1$

Coefficient of z^{-k} in 2nd term = 3^{k-1} ; $k \geq 1$

Coefficient of z^{k} in 3rd term = $\frac{-1}{4^{k+1}}$; $k \geq -1$

Coefficient of z^k in 3^{rd} term = $\frac{-1}{4^{-k+1}}$; $k \le 0$

Hence $z^{-1}[f(z)] = 2^{k-1} + 3^{k-1}$ **; $k \ge 1$**

$$= \frac{-1}{4^{-k+1}} \;;\; k \le 0$$

Q3] c) Solve the differential equation:-

$\frac{d^2y}{dt^2}$ **+ 4y = f**(t) ; f(t) **= H**(t-2) ;y(0) **= 0;** $y'(0)$ **= 1** **(8)**

Using Laplace transform

Solution:-

Let $\bar{y}$ be the Laplace transform of y $L[y] = \bar{y}$

Taking Laplace on both sides

$L[y''] + L[4y] = L[f(t)]$

$S^2 \bar{y} + Sy(0) - y'(0) + 4\bar{y} = L[f(t)]$

$S^2 \bar{y} + 0 - 1 + 4\bar{y} = L[f(t)]$

$S^2 \bar{y} - 1 + 4\bar{y} = L[f(t)]$

$(S^2 + 4)\bar{y} = 1 + L[f(t)]$

$(S^2 + 4)\bar{y} = 1 + L[H(t-2)]$

$$\bar{y} = \frac{1}{S^2+4} + \frac{e^{-2s}}{s(S^2+4)}$$

$$\bar{y} = \frac{1}{S^2+4} + \left[\frac{1}{s} - \frac{1}{S^2+4}\right]\frac{e^{-2s}}{4}$$

Taking inverse on both sides

$$y = L^{-1}\left(\frac{1}{S^2+4}\right) + L^{-1}\left[\frac{e^{-2s}}{4}\left(\frac{1}{s}\right)\right] - L^{-1}\left[\frac{e^{-2s}}{4(S^2+4)}\right]$$

$$\mathbf{y = \frac{\sin 2t}{2} + \frac{1}{4}H(t-2) - \frac{1}{4}\cos 2(t-2)H(t-2)}$$

Q4] a) Find $Z\{f(k)\times g(k)\}$ if $(k) = \left(\frac{1}{2}\right)^k$; $g(k) = \cos\pi k$ (6)

Solution:-

$$Z\left\{\left(\frac{1}{2}\right)^k \times \cos\pi k\right\}$$

$$Z\left\{\left(\frac{1}{2}\right)^k\right\} = \sum_{k=0}^{\infty}\frac{1}{2^k}\times Z^{-k} = \sum_{k=0}^{\infty}\frac{1}{2Z^k}$$

$$Z\left\{\left(\frac{1}{2}\right)^k\right\} = 1 + \frac{1}{2Z} + \frac{1}{(2Z)^2} + \frac{1}{(2Z)^3} + - - -$$

$$Z\left\{\left(\frac{1}{2}\right)^k\right\} = \frac{2Z}{2Z-1}$$

$$Z\{\cos\pi k\} = \sum_{k=0}^{\infty}\cos\pi k\times Z^{-k}$$

$$Z\{\cos\pi k\} = \frac{Z(Z-\cos\pi)}{Z^2-2Z\cos\pi+1} = \frac{Z(Z-(-1))}{Z^2-2Z(-1)+1} = \frac{Z(Z+1)}{Z^2+2Z+1} = \frac{Z}{Z+1}$$

$$\mathbf{Z\{\cos\pi k\} = \frac{Z}{Z+1}}$$

By convolution Theorem; $Z\{f(k)\times g(k)\} = \left(\frac{2Z}{2Z-1}\right)\left(\frac{Z}{Z+1}\right)$

Q4] b) Find the Sperman's Rank Correlation Coefficient b/w X and Y (6)

X	60	30	37	30	42	37	55	45
Y	50	25	33	27	40	33	50	42

Solution:-

X	R_1	Y	R_2	$R_1 - R_2$	D^2 $(R_1-R_2)^2$
60	8	50	7.5	-0.5	0.25
30	1.5	25	1	0.5	0.25
37	3.5	33	3.5	0	0
30	1.5	27	2	-0.5	0.25
42	5	40	5	0	0
37	3.5	33	3.5	0	0
55	7	50	7.5	-0.5	0.25
45	6	42	6	0	0
					$\sum = 1$

For repeated ranks;

$$R = 1 - \frac{6\left\{\sum D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \text{------} \ \frac{1}{12}(m_4^3 - m_4)\right\}}{8^3 - 8}$$

$$R = 1 - \frac{6\left\{1 + \frac{1}{12}(8-2) + \frac{1}{12}(8-2) + \frac{1}{12}(8-2) + \frac{1}{12}(8-2)\right\}}{8^3 - 8}$$

R = 0.9643

Q4] c) Find inverse Laplace transform of **(8)**

1) $\dfrac{3S+1}{(s+1)^4}$ 2) $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$

Solution:-

1) $\frac{3S+1}{(s+1)^4}$

By first shifting theorem of replace inverse;

$$L^{-1}\left[\frac{1}{(s+a)^n}\right] = e^{-at}L^{-1}\left[\frac{1}{(s)^n}\right]$$

$$L^{-1}\left[\frac{3S+1}{(s+1)^4}\right] = e^{-t}L^{-1}\left[\frac{3(s-1)+1}{(s+1-1)^n}\right] = e^{-t}L^{-1}\left[\frac{3s-2}{(s)^n}\right] = e^{-t}L^{-1}\left[\frac{3s}{(s)^3}-\frac{2}{(s)^4}\right]$$

$$L^{-1}\left[\frac{3S+1}{(s+1)^4}\right] = e^{-t}\left[\frac{3t^2}{2!}-\frac{2t^3}{3!}\right]$$

$$L^{-1}\left[\frac{3S+1}{(s+1)^4}\right] = e^{-t}\left[\frac{3t^2}{2!}-\frac{2t^3}{3!}\right]$$

2) $\frac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$

$$\frac{e^4.e^{-3s}}{(s+4)^{5/2}} = e^4L^{-1}\left[\frac{e^{-3s}}{(s+4)^{5/2}}\right]$$

Here $\varphi(s) = \frac{1}{(s+4)^{5/2}}$ and $a = 3$

$$L^{-1}[\varphi(s)] = L^{-1}\left[\frac{1}{(s+4)^{5/2}}\right] = e^{-4t}L^{-1}\left[\frac{1}{(s)^{5/2}}\right] = e^{-4t}\frac{t^{3/2}}{\sqrt{5/2}} = \frac{e^{-4t}.\ t^{3/2}}{3/2\times1/2\times\sqrt{1/2}}$$

$$L^{-1}[\varphi(s)] = \frac{e^{-4t}.\ t^{3/2}.4}{3\sqrt{\pi}}$$

$$L^{-1}\left[\frac{e^{-as}}{(s+4)^{5/2}}\right] = f(t-a)H(t-a) = \frac{4}{3\sqrt{\pi}}\times e^{-4(t-3)}(t-3)^{\frac{3}{2}}H(t-3)$$

$$L^{-1}\left[\frac{e^4.e^{-3s}}{(s+4)^{5/2}}\right] = e^4\times\frac{4}{3\sqrt{\pi}}\times e^{-4(t-3)}(t-3)^{\frac{3}{2}}H(t-3)$$

Q5] a) Find inverse Laplace Transform using convolution theorem; (6)

$$\frac{1}{(s-4)^2(s+3)}$$

Solution:-

Let $\varphi_1(s) = \frac{1}{s+3}$ and $\varphi_2(s) = \frac{1}{(s-4)^2}$

$L^{-1}[\varphi_1(s)] = e^{-3t}$ and $L^{-1}[\varphi_2(s)] = e^{4t}L^{-1}\left[\frac{1}{s^2}\right] = e^{4t}t$

$L^{-1}[\varphi_1(s)] = \int_0^t e^{-3u}.e^{4(t-u)}(t-u)du = \int_0^t e^{(4t-7u)}(t-u)du$

$L^{-1}[\varphi_1(s)] = e^{4t}\int_0^t e^{-7u}(t-u)du = e^{4t}\left[(t-u)\frac{e^{-7u}}{-7}-\frac{(-1)e^{-7u}}{49}\right]_0^t$

$L^{-1}[\varphi_1(s)] = e^{4t}\left[\frac{e^{-7t}}{49}+\frac{t}{7}+\frac{1}{49}\right] = e^{4t}\left[\frac{t}{7}+\frac{e^{-7t}-1}{49}\right]$

$$L^{-1}[\varphi_1(s)] = e^{4t}\left[\frac{t}{7}+\frac{e^{-7t}-1}{49}\right]$$

Q5] b) Show that the functions $f_1(x) = 1$; $f_2(x) = x$ are orthogonal on (-1,1); determine the constant a,b such that the function $f(x) = -1 + ax + bx^2$ is orthogonal to both $f_1(x)$,$f_2(x)$ on the (-1,1). (6)

Solution:-

$f_1(x) = 1$; $f_2(x) = x$; $f_3(x) = 1 + ax + bx^2$

Case 1:- $m \neq n$

$\int_{-1}^{1}[f_1(x)]^2dx = \int_{-1}^{1}1dx = [x]_{-1}^{1} = 1-(-1) = 2 \neq 0$

$\int_{-1}^{1}[f_2(x)]^2dx = \int_{-1}^{1}x^2dx = [\frac{x^3}{3}]_{-1}^{1} = \frac{1}{3}-\left(\frac{-1}{3}\right) = \frac{2}{3} \neq 0$

$f_1(x)$ & $f_2(x)$ are orthogonal in [-1,1]

$f_3(x)$ is orthogonal with $f_1(x)$

$\int_{-1}^{1}[f_1(x)\times f_3(x)]dx=0$

$\int_{-1}^{1}[-1+ax+bx^2]dx=0$

$\int_{-1}^{1}[-1]dx+\int_{-1}^{1}[ax]dx+\int_{-1}^{1}[bx^2]dx=0$

$-(1-(-1)) + a\left[\frac{x^2}{2}\right]_{-1}^{1} + b\left[\frac{x^3}{3}\right]_{-1}^{1} = 0$

$-2 + 0 + b\left[\frac{1}{3}+\frac{1}{3}\right] = 0$

$-2 + \frac{2b}{3} = 0$

$b = 3$

Also $f_3(x)$ is orthogonal with $f_2(x)$

$\int_{-1}^{1}[f_2(x)\times f_3(x)]dx=0$

$\int_{-1}^{1}x[-1+ax+bx^2]dx=0$

$\int_{-1}^{1}[-x+ax^2+bx^3]dx=0$

$\left[\frac{-x^2}{2}+\frac{ax^3}{3}+\frac{bx^4}{4}\right]_{-1}^{1} = 0$

$\left(\frac{-1}{2}+\frac{a}{3}+\frac{b}{4}\right)-\left(\frac{-1}{2}-\frac{a}{3}+\frac{b}{4}\right) = 0$

$\frac{2a}{3} = 0$

$a = 0$

Ans :- a = 0 and b = 3

Q5] c) Find the Laplace transform of:-

1) $e^{-3t}\int_0^t t\sin 4t\,dt$ **2)** $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$ **(8)**

Solution:-

1) $e^{-3t}\int_0^t t\sin 4t\,dt$

$$L[\sin 4t] = \frac{4}{s^2+16} = \varphi(s)$$

$$L[\sin 4t] = \frac{(-1)d[\ \varphi(s)]}{ds} = \frac{(-1)d\left[\frac{4}{s^2+16}\right]}{ds}$$

$$-4\frac{d}{ds}\left[\frac{1}{s^2+16}\right] = -4\left[\frac{(3^2+16)0-1(2s)}{(s^2+16)^2}\right]$$

$$-4\left[\frac{-2s}{(s^2+16)^2}\right] = \frac{8s}{(s^2+16)^2}$$

$$L\left[\int_0^t t\sin 4t\,dt\right] = \frac{1}{s}\times\frac{8s}{(s^2+16)^2} = \frac{8}{(s^2+16)^2}$$

$$L\left[e^{-3t}\int_0^t t\sin 4t\,dt\right] = \frac{8}{[(s+3)^2+16]^2}$$ ....(by first shifting method)

$$L\left[e^{-3t}\int_0^t t\sin 4t\,dt\right] = \frac{8}{[s^2+6s+9+16]^2} = \frac{8}{[s^2+6s+25]^2}$$

$$\mathbf{L}\left[e^{-3t}\int_0^t t\sin 4t\,dt\right] = \frac{8}{[s^2+6s+25]^2}$$

2) $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$

$$L[e^{-t}-e^{-2t}] = \frac{1}{s+1}-\frac{1}{s+2} = \varphi(s)$$

$$L\left[\frac{e^{-t}-e^{-2t}}{t}\right] = \int_s^\infty \varphi(s)$$ .... [division by t]

$$\int_s^\infty \frac{1}{s+1}-\frac{1}{s+2}$$

$$[\ln(s+1)-\ln(s+2)]_s^\infty = \left[\ln\left(\frac{s+1}{s+2}\right)\right]_s^\infty$$

$$\left[\ln\left(\frac{\frac{1}{s}+1}{\frac{2}{s}+1}\right)\right]_s^{\infty} = \left[\ln(0)-\ln\left(\frac{s+1}{s+2}\right)\right] = -\ln\left(\frac{s+1}{s+2}\right) = \ln\left(\frac{s+2}{s+1}\right)\ \int_0^\infty e^{-st}\times\frac{e^{-t}-e^{-2t}}{t}dt = \ln\left(\frac{s+2}{s+1}\right)$$

Put s = 0

$$\int_0^{\infty} \frac{e^{-t}-e^{-2t}}{t} dt = \ln(2)$$

Q6] a) Fit a second degree parabola to the given data (6)

X	1	1.5	2	2.5	3	3.5	4
Y	1.1	1.3	1.6	2	2.7	3.4	4.1

Solution:-

Sr	x	y	x^2	x^3	x^4	xy	x^2y
1	1	1.1	1	1	1	1.1	1.1
2	1.5	1.3	2.25	3.375	5.0625	1.95	2.925
3	2	1.6	4	8	16	3.2	6.4
4	2.5	2	6.25	15.625	39.062	5	12.5
5	3	2.7	9	27	81	8.1	24.3
6	3.5	3.4	12.25	42.875	150.06	11.9	41.65
7	4	4.1	16	64	256	16.4	65.6
$\sum$	17.5	16.2	50.75	161.875	548.1845	47.65	154.475

The normal equation are:

$\sum y = Na + b\sum x + c\sum x^2$

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4$

16.2 = 7a + b(17.5) + c(50.75)

$47.65 = 17.5a + b(50.75) + c(161.875)$

$154.475 = 50.75a + b(161.875) + c(548.1845)$

Solving simultaneously;

$a = 0.8329 \quad b = 2.4091 \times 10^{-4} \quad c = 0.2042$

$$\mathbf{y = 0.8329 + 2.4091 \times 10^{-4} x + 0.2042 x^2}$$

Q6] b) Find the image of $\left|z-\frac{5}{2}\right| = \frac{1}{2}$ under the transformation $\omega = \frac{3-z}{z-2}$ (6)

Solution:-

$\omega = \frac{3-z}{z-2}$

$\omega(z-2) = (3-z)$

$\omega z - 2\omega = 3 - z$

$\omega z + z = 3 + 2\omega$

$z(1+\omega) = 3 + 2\omega$

$z = \frac{3+2\omega}{(1+\omega)}$

$\left|\frac{3+2\omega}{(1+\omega)} - \frac{5}{2}\right| = \frac{1}{2}$ i.e. $\left|\frac{6+4\omega-5-5\omega}{2(1+\omega)}\right| = \frac{1}{2}$

$\left|\frac{1-\omega}{2+2\omega}\right| = \frac{1}{2}$ i.e. $\left|\frac{1-(u-iv)}{2+2(u+iv)}\right| = \frac{1}{2}$

$\left|\frac{(1-u)-iv)}{(2+2u)+2iv}\right| = \frac{1}{2}$

$\frac{(1-u)^2+v^2}{(2+2u)^2+4v^2} = \frac{1}{4}$

$4[(1-u)^2+v^2] = (2+2u)^2 + 4v^2$

u = 0

Imaginary axis

Q6] c) Find half range cosine series for f(x) = xsinx and hence find (8)

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\dots\dots = \frac{\pi-2}{4}$$

Solution:-

F(x) = xsinx

$$a_0 = \frac{1}{\pi}\int_0^x f(x)dx \quad = \frac{1}{\pi}\int_0^x xsinxdx$$

$$a_0 = \frac{1}{\pi}[x(-cosx)-(-sinx)]_0^\pi$$

$$a_0 = \frac{1}{\pi}[\pi(-(-1))] = 1 \quad \dots\dots\dots\dots\dots(1)$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)cosnxdx \quad = \frac{2}{\pi}\int_0^\pi xsinxcosnxdx$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\left\{\frac{1}{2}[sin(n+1)x+sin(n-1)x]\right\}dx$$

$$a_n = \frac{1}{\pi}\int_0^\pi x\{sin(n+1)x+sin(n-1)x\}dx$$

$$a_n = \frac{1}{\pi}\left[x\left\{\frac{-cos(n+1)x}{n+1} - \frac{cos(n-1)x}{n-1}\right\} - \left\{\frac{-sin(n+1)x}{(n+1)^2} - \frac{sin(n-1)}{(n-1)^2}\right\}\right]_0^\pi$$

$$a_n = \frac{1}{\pi}\left[\pi\left\{\frac{cosn\pi}{n+1} - \frac{cosn\pi}{n-1}\right\}\right] = \frac{1}{\pi}\left[\pi\left\{\frac{-1^n}{n+1} - \frac{-1^n}{n-1}\right\}\right]$$

$$a_n = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1} \qquad \text{for } n \neq 1$$

For n=1 put n=1 in equation (1)

$$a_1 = \frac{2}{\pi}\int_0^\pi xsinxcosnxdx$$

$$a_1 = \frac{1}{\pi}\int_0^\pi 2xsinxcosnxdx$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$a_1 = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(\frac{-1}{2}\right)-0\right] = \frac{-1}{2}$$

$$x\sin x = a_0 + \sum_{n=0}^{\infty} a_n \cos nx = 1 + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

Put $x = \pi/2$

$$\frac{\pi}{2} = 1 - 0 + 2\left[\sum\frac{(-1)^3}{3}\cos 2\left(\frac{\pi}{2}\right)+\frac{(-1)^4}{8}\cos\frac{3\pi}{2}\ldots\ldots\ldots\right]$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{1}{3}-\frac{1}{15}+\frac{1}{35}\ldots\ldots\ldots\right]$$

$$\frac{\pi-2}{4} = \left[\frac{1}{3}-\frac{1}{15}+\frac{1}{35}\ldots\ldots\ldots\right]$$
