3 Hours] **[Total Marks: 100**

N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

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1. Choose the correct option. Attempt all the subquestions.

(i) Let H be a normal subgroup of G. Let $\circ(aH) = 3$ in $\frac{G}{H}$ and $|H| = 10$, then order of a is (2)

(a) 1
(b) 30
(c) one of 3, 6, 15 or 30
(d) None of these.

(ii) Which of the following is not true for a normal subgroup H of a group G? (2)

(a) $aHa^{-1} \subseteq H$ for each $a \in G$.
(b) $aHa^{-1} = H$ for each $a \in G$.
(c) Every left coset of H in G is also a right coset of H in G i.e. $aH = Ha$ for each $a \in G$.
(d) G/H is Abelian.

(iii) Which of the following is not true? (2)

(a) $\mathbb{Z}_3$ is isomorphic to A_3.
(b) $\mathbb{Z}_4$ is isomorphic to $<(2\ 1\ 3\ 4)>$, a subgroup of S_4.
(c) V_4 is isomorphic to $\{I, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ a subgroup of S_4.
(d) $\mathbb{Z}_6$ is isomorphic to a subgroup of A_4.

(iv) The group of units of a ring is (2)

(a) Abelian but may not be cyclic
(b) Cyclic
(c) may not be Abelian
(d) finite

(v) Consider the ideals of ring of integers $I = 6\mathbb{Z}$ and $J = 10\mathbb{Z}$, then (2)

(a) $I + J = 22\mathbb{Z}$, $IJ = 120\mathbb{Z}$.

(b) $I + J = 2\mathbb{Z}$, $IJ = 60\mathbb{Z}$.

(c) $I + J = 2\mathbb{Z}$, $IJ = 30\mathbb{Z}$.

(d) None of these.

(vi) In the polynomial ring $\mathbb{Z}[x]$, consider $I = \{f(x) : f(0) = 0\}$, then (2)

(a) I is an ideal.

(b) I is a maximal ideal.

(c) I is ideal but neither prime ideal nor maximal.

(d) I is prime ideal but not maximal ideal.

(vii) Which of the following is true in $\mathbb{Z}[\sqrt{-5}]$ (2)

(a) $2 + \sqrt{-5}$ is irreducible but not prime.

(b) $2 + \sqrt{-5}$ is prime.

(c) 3 is prime.

(d) $2 + \sqrt{-5}$ is reducible.

(viii) The number of maximal ideals in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is (2)

(a) 1. (b) 3.

(c) 6. (d) 9.

(ix) The field of quotients of $\mathbb{Z}[i]$ is (2)

(a) $\mathbb{Q}[i]$

(b) $\mathbb{R}$

(c) $\mathbb{C}$

(d) None of these.

(x) Let $I = (x^2 + x + 1)$ in $\mathbb{Z}_n[x]$, $1 \leq n \leq 10$ Then, $\mathbb{Z}_n[x]/I$ is a field if (2)

(a) $n = 3$

(b) for all $n \leq 5$

(c) $n = 7$

(d) $n = 2, 5$

2. (a) Answer any **ONE**

(i) Let G and G' be groups and $f : G \to G'$ be an onto homomorphism. Prove that if H is a subgroup G then $f(H) = \{f(h) : h \in H\}$ is a subgroup of G' and $f(Ha) = f(H)f(a)$ for each $a \in G$. Further, if H is normal in G then $f(H)$ is normal in G'. Give example to show that $f(H)$ need not be normal in G' if f is not onto. (8)

(ii) If $a \in G_1, b \in G_2$ such that $\circ(a) = m, \circ(b) = n$, then prove that $(a, b)^k = (a^k, b^k)$ for every $k \in \mathbb{N}$ and $\circ(a, b) = lcm(m, n)$. Hence prove that, G_1, G_2 are cyclic then $G_1 \times G_2$ is cyclic if and only if $\circ(G_1)$ and $\circ(G_2)$ are relatively prime. (8)

(b) Answer any **TWO**

(i) Show that kernel of a group homomorphism $f : G \to G'$ is a normal subgroup of G. Also show that for any normal subgroup H of G there is a group homomrphism $\eta : G \to G/H$ such that $ker\ \eta = H$. (6)

(ii) If $G/Z(G)$ is cyclic then prove that G is Abelian. (6)

(iii) Show that order of each element of the quotient group $\dfrac{\mathbb{Q}}{\mathbb{Z}}$ is finite. (6)

(iv) Show that $\{e, b\}$ is normal in $\{e, b, a^2b, a^2\}$ but not normal in $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $a^4 = e = b^2, aba = b$. (6)

3. (a) Answer any **ONE**

(i) Define characteristic of a ring R. Show that, characteristic of a ring R is n if and only if the order of the multiplicative identity of R is n in the group $(R, +)$. Give example of an infinite ring with characteristic 2. (8)

(ii) Let R be a commutative ring. If I, J are ideals in R, Show that $I \cap J$, $I + J$ and IJ are ideals of R, where (8)

$I+J = \{x+y\ :\ x \in I, y \in J\}$ and $IJ = \left\{ \sum_{i=1}^{n} x_i y_i\ :\ x_i \in I, y_i \in J, n \in \mathbb{N} \right\}$.

Further if $R = I + J$, show that $I \cap J = IJ$.

(b) Answer any **TWO**

(i) Let A be a subring and B be an ideal of a ring R. Then prove that $A \cap B$ is an ideal of A and $A/(A \cap B) \simeq (A + B)/B$. (6)

(ii) Let R, R' be commutative rings and $f : R \to R'$ be a ring homomorphism. Show that- (6)

(I) If f is surjective, I is an ideal of R, then $f(I)$ is an ideal of R'.

(II) If I' is an ideal of R' , then $f^{-1}(I')$ is an ideal of R.

(iii) Let $R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ and $I = \{a + b\sqrt{2} : a, b \in \mathbb{Z}, a \text{ is even}\}$. Show that the quotient ring R/I is isomorphic to $\mathbb{Z}_2$. (6)

(iv) Let R be a commutative ring with prime characteristic p and $f : R \to R$ be defined as $f(a) = a^p$ for $a \in R$. Show that f is a ring homomorphism. (6)

4. (a) Answer any **ONE**

   (i) Show that an ideal P in a commutative ring R is a prime ideal if and only if the quotient ring R/P is integral domain. (8)

   Further prove that in a finite commutative ring every prime ideal is maximal.

   (ii) Define irreducible polynomial. (8)

   Let F be a field. Show that $F[x]/ < f(x) >$ is a field if and only if $f(x)$ is irreducible over F.

   (b) Answer any **TWO**

   (i) Let R, S be commutative rings. And $f : R \to S$ be an onto ring homomorphism. Prove that, if M is a maximal ideal in S then, $f^{-1}(M)$ is a maximal ideal in R. (6)

   (ii) Show that the only irreducible polynomials in $\mathbb{R}[x]$ are a linear polynomial $x-a$ or quadratic polynomial x^2+bx+c such that $b^2-4c < 0$, where $a, b, c \in \mathbb{R}$. (6)

   (iii) Show that in $\mathbb{Z}[i]$, 3 is irreducible but 2 is not irreducible. (6)

   (iv) Show that $< x, 2 >$, the ideal generated by x and 2 is a maximal ideal of $\mathbb{Z}[x]$. Further show that this ideal is not principal ideal. (6)

5. Answer any **FOUR**

   (a) Let G be a group anf H be a normal subgroup of G. Then prove that (5)

   (p) $(Ha)^n = Ha^n$ for all $n \in \mathbb{Z}$.

   (q) $\circ(Ha)$ divides $\circ(a)$.

   (b) Find a subgroup of order 9 in $\mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_{15}$. (5)

   (c) Show that a finite field of size 8 has characteristic 2. (5)

   (d) Determine all the ideals of $\mathbb{R}[x]/(x^3 + 3x^2 - 4)$ by stating the results used. (5)

   (e) Let R be commutative and I, J be ideal of R and P is a prime ideal of R that contains $I \cap J$. Prove that either $I \subseteq P$ or $J \subseteq P$. (5)

   (f) Let $\mathbb{F}$ be a field. Show that every ideal of $\mathbb{F}[x]$ is a principal ideal. (5)

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