Duration: 3 Hrs Marks: 100

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. Choose correct alternative in each of the following: (20)

(i) Let $A = \{x \in \mathbb{R} : |\sin x| \leq 1/2\}$, with usual metric on $\mathbb{R}$, which of the following statements is true?
(a) A is an open subset of $\mathbb{R}$.
(b) A is a closed subset of $\mathbb{R}$.
(c) A is an open as well as closed subset of $\mathbb{R}$.
(d) None of these

(ii) Which of the following subset of usual metric space $\mathbb{R}$ is not dense?
(a) $\mathbb{Q}$ (b) $\mathbb{R} \setminus \mathbb{Q}$ (c) $\mathbb{N}$ (d) $\mathbb{R}$

(iii) Let $(\mathbb{R}, d)$ be a metric space where d is a discrete metric. Then, which of the following subset of $(\mathbb{R}, d)$ is infinite?
(a) $B(0, 0.5)$ (b) $B(0, 1)$ (c) $B(0, 2)$ (d) None of these.

(iv) Which of the following sequences in $(\mathbb{Q}, d)$, d is a usual metric from $\mathbb{R}$, is convergent in $\mathbb{Q}$?
(a) $x_n = \left(1 + \frac{1}{n}\right)^n, n \in \mathbb{N}$
(b) $x_n = \frac{2n+1}{3n+2}, n \in \mathbb{N}$.
(c) $x_n = \frac{\lfloor \sqrt{2}n \rfloor}{n}, n \in \mathbb{N}$.
(d) $x_n = \frac{n^2+1}{n+3}, n \in \mathbb{N}$.

(v) Every Cauchy sequence is eventually constant in
(a) $(\mathbb{N}, d)$ where d is usual.
(b) $(\mathbb{Q}, d)$ where d is usual.
(c) $(\mathbb{R} \setminus \mathbb{Q}, d)$ where d is usual.
(d) None of the above.

(vi) Let d_1 and d_2 be metrics on X such that $k_1 d_2(x, y) \leq d_1(x, y) \leq k_2 d_2(x, y)$ for all $x, y \in X$ where $k_1, k_2 > 0$ are constants. Then the statement which is not true is
(a) (x_n) is Cauchy in (X, d_1) if and only if (x_n) is Cauchy in (X, d_2).
(b) $x_n \longrightarrow p$ in (X, d_1) if and only if $x_n \longrightarrow p$ in (X, d_2).
(c) (x_n) is bounded in (X, d_1) if and only if (x_n) is bounded in (X, d_2).
(d) None of the above.

(vii) In $\mathbb{R}$ with respect to usual distance $\cap_{n \in \mathbb{N}} F_n$ is a singleton set when
(a) $F_n = [-n, n]$ (b) $F_n = [n, n+1]$ (c) $F_n = [1 - \frac{1}{n}, 1]$ (d) $F_n = [0, n]$

(viii) Which of the following subset of $\mathbb{R}$ or $\mathbb{R}^2$ is compact with respect to the Euclidean metric?

(a) $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$ (c) $\{x \in \mathbb{Z} : x^2 < 2\}$
(b) $\{x \in \mathbb{R} : x^2 < 2\}$ (d) $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$

(ix) Let A be a compact subset of $\mathbb{R}$. Then
(a) $\overline{A}$ may not be compact. (b) A° may not be compact.
(c) ∂A may not be compact. (d) None of the above.

(x) Let (X,d) be a metric space and (x_n) be a sequence in X such that $x_n \to x_0$ as $n \to \infty$. Then
(a) $\{x_n : n \in \mathbb{N}\}$ is a compact subset of X
(b) $\{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is a compact subset of X
(c) $\{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is compact only if (x_n) is a sequence of distinct points.
(d) None of the above.

2. (a) Attempt any One of the following: (8)
(i) Every infinite bounded subset of $\mathbb{R}$ has a limit point. (distance being usual)
(ii) Let (X,d) be a metric space. Prove the following:
(I) Arbitrary union of open sets is open.
(II) A subset G of X is open if and only if it is an union of open balls.

(b) Attempt any Two of the following: (12)
(i) Let A be a subset of a metric space (X,d) . Prove that
(I) $\overline{(X \setminus A)} = X \setminus A^\circ$
(II) $(X \setminus A)^\circ = X \setminus (\overline{A})$
(ii) Let (X,d) be a metric space. $d_1 : X \times X \to \mathbb{R}$ is a metric defined as $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$, $\forall x,y \in X$. Show that d and d_1 are equivalent metrics on X
(iii) Let $(X, \|\ \|)$ be a normed linear space and $A \neq \emptyset, A \subseteq X$. Show that if $U \neq \emptyset, U \subseteq X$ is an open set then $U + A$ is open.
(iv) Show that $B_1(0,1)$ in $(C[0,1], \|\ \|_1)$ is open in $(C[0,1], \|\ \|_\infty)$ where $\|f\|_1 = \displaystyle\int_0^1 |f(t)|dt$, $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$ and $B_1(0,1) = \{f \in C[0,1] : \|f\|_1 < 1\}$.

3. (a) Attempt any One of the following: (8)
(i) Let (X,d) be a metric space and A be a subset of X. Show that $p \in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p.
(ii) State and prove the Nested interval theorem in $\mathbb{R}$.

(b) Attempt any Two of the following: (12)
(i) If (x_n) and (y_n) are sequences in a metric space (X,d) such that $x_n \longrightarrow p$ and $y_n \longrightarrow q$ then show that the sequence of real numbers $d(x_n, y_n) \longrightarrow d(p,q)$ in $(\mathbb{R}, \text{usual})$.
(ii) Let (X,d) be a metric space and Y be a non-empty subset of X. Prove that a subset G of Y is open in the subspace (Y,d) if and only if $G = V \cap Y$ where V is an open set in (X,d).

(iii) Check if Cantor's Theorem is applicable in the following examples. Also, find $\cap_{n\in\mathbb{N}}F_n$ in each case, where (F_n) is a sequence of subsets of $X \subseteq \mathbb{R}$.

(i) $X = [-1, 1]$ and distance d is the usual distance, $F_n = [-\frac{1}{n}, \frac{1}{n}]$

(ii) $X = \mathbb{R}, d$ discrete metric, $F_n = (0, \frac{1}{n})$

(iv) Show that $(\mathbb{N}, d)$ is a complete metric space where for $m, n \in \mathbb{N}$,

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ 1 + \dfrac{1}{m+n} & \text{if } m \neq n \end{cases}$$

4. (a) Attempt any One of the following: (8)

(i) Let A be a non-empty subset of the metric space $(\mathbb{R}, d)$ where d is the usual metric. Prove that A is sequentially compact if and only if A satisfies the Bolzano-Weierstrass property.

(ii) Show that a compact subset of a metric space is closed and bounded. Give an example to show that a closed and bounded subset need not be compact.

(b) Attempt any Two of the following: (12)

(i) Suppose (X, d) is a metric space and $\mathcal{C}$ is a non-empty collection of compact subsets of X then show that if $\mathcal{C}$ is finite then $\bigcup\limits_{K\in\mathcal{C}} K$ is a compact subset of X.

(ii) Show that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a compact subset of $\mathbb{R}^2$, distance being Euclidean.

(iii) If $X = [0, 1] \subset (\mathbb{R}, d)$, where d is the discrete metric, show that the open cover $\left\{B(x, \frac{1}{2}) : x \in [0, 1]\right\}$ of X has no finite subcover.

(iv) Show that $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + 2x_2^2 + \cdots + nx_n^2 \leq (n+1)^2\}$ is a compact subset of $(\mathbb{R}^n, d), d$ being Euclidean.

5. Attempt any Four of the following: (20)

(a) State and prove Hausdorff property in a metric space (X, d).

(b) Show that $S = \{x \in \mathbb{Q} : 3 < x^2 < 5\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual metric.

(c) $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous such that f takes only rational values then show that f is a constant function.

(d) Prove or disprove : If d_1 and d_2 are equivalent metrics on X and (X, d_1) is a complete metric space then (X, d_2) is also a complete metric space.

(e) If A, B are compact subsets of $\mathbb{R}$ with respect to usual distance, show that $A \times B$ is a compact subset of $\mathbb{R}^2$ with Euclidean metric.

(f) Consider the set $A = [0, 1]$ in the metric space $(\mathbb{R}, d), d$ being the discrete metric. Show that the open cover $\{B(x, \frac{1}{2})\}_{x\in[0,1]}$ of A has no finite subcover.