Duration 3 Hrs | Marks: 100

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. Choose correct alternative in each of the following: (20)

i. Which of the following maps $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a metric on $\mathbb{R}^2$, where d is defined as, for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$
(a) $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$ (b) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
(c) $d(x, y) = |x_2 - y_2|$ (d) $d(x, y) = (x_1 - y_1)^2$

ii. Consider the discrete metric d defined on a non-empty subset X containing at least two elements. Then, for $p \in X$, which of the following is false?
(a) $B(p, 2) = B(p, 3)$ (b) $B(p, 1) \subset B(p, 2)$
(c) $B(p, 0.5) = B(p, 1)$ (d) $B(p, 1) = B(p, 2)$

iii. Which of the following sets is bounded in $\mathbb{R}$?
(a) $A = \mathbb{R}$ with respect to the usual metric
(b) $A = \mathbb{R}$ with respect to the discrete metric
(c) $A = [a, \infty)$ with respect to the usual metric
(d) None of the above

iv. Let $A = \mathbb{Q}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $C = \bigcap_{n \in \mathbb{N}} (\frac{-1}{n}, \frac{1}{n})$ be subsets of $\mathbb{R}$(distance being usual), then,
(a) $A^\circ = B^\circ = C^\circ = \emptyset$ (b) $\overline{A} = \mathbb{R}; \overline{B} = B; C^\circ = \emptyset$
(c) $A^\circ = \emptyset; B^\circ = B; C^\circ = \emptyset$ (d) $\overline{A} = \mathbb{R}; B^\circ = \emptyset; C^\circ = \{0\}$

v. $f : [0, 1] \to [0, 1]$ is defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 - x & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$, then
(a) f is continuous on $[0, 1]$ and does not satisfy intermediate value property.
(b) f satisfies intermediate value property but f is not continuous.
(c) f is continuous only at $x = 1/2$ and $f([0, 1]) = [0, 1]$.
(d) none of the above.

vi. Every Cauchy Sequence is eventually a constant in
(a) $(\mathbb{N}, d)$, where d is the usual distance. (b) $(\mathbb{Q}, d)$, where d is the usual distance.
(c) $(\mathbb{R} \setminus \mathbb{Q}, d)$, where d is the usual distance. (d) None of the above.

vii. Let (X, d) be a complete metric space , A and B are complete subspaces of (X, d) and $A \cap B$ is nonempty then
(a) $A \cup B$ is complete and $A \cap B$ is not. (b) $A \cap B$ is complete and $A \cup B$ is not.
(c) $A \cup B$ and $A \cap B$ are complete. (d) none of the above.

viii. In $(\mathbb{R}, d)$, where d is usual metric,
(a) $\{1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{-1, 0, 1\}$ is compact. (b) $\{1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$ is not compact.
(c) $\{1 + \frac{1}{n} : n \in \mathbb{N}\}$ is compact. (d) None of these.

ix. In the metric space $(\mathbb{Z}, d)$, ($\mathbb{Z}$ is the set of integers, d is usual distance), $K \subset \mathbb{Z}$ is compact
(a) if and only if K is closed. (b) if and only if K is bounded.
(c) if and only if K has a limit point. (d) if and only if $0 \in K$.

x. Let (x_n) be a sequence in $[0,1]$ with usual metric from $\mathbb{R}$. Then, which of the following is **not true**?
(a) (x_n) has a convergent subsequence.
(b) (x_n) is bounded but may not be convergent.
(c) (x_n) is Cauchy.
(d) (x_n) may have subsequences converging to different limits.

2. (a) Attempt any One of the following: (8)
   (i) Define an open ball $B(x,r)$ in a metric space (X,d) and show that every open ball is an open set. Also give an example to show that the converse need not be true.
   (ii) Let (X,d) be a metric space and $A, B \subseteq X$. Show that
      (I) $A \subseteq B \Longrightarrow A^\circ \subseteq B^\circ$
      (II) $(A \cap B)^\circ = A^\circ \cap B^\circ$
      (III) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ and the inequality may be strict.

   (b) Attempt any Two of the following: (12)
   (i) State and prove Hausdorff property in a metric space (X,d).
   (ii) Let (X,d) be a metric space and $A \subseteq X$. Show that $\overline{A}$ is a closed set and it is the smallest closed set containing A.
   (iii) Prove that $(\mathbb{Z}, d)$ and $(\mathbb{Z}, d_1)$ where d is the usual distance (induced from $\mathbb{R}$) and d_1 is the discrete metric in $\mathbb{Z}$, are equivalent metric spaces.
   (iv) Answer the following:
      (I) Consider the subspace $Y = [0, \infty)$ of $\mathbb{R}$ where distance d in $\mathbb{R}$ is usual. Find $B_Y(0,5)$ an open ball in the subspace (Y,d).
      (II) In $(\mathbb{R}, \text{usual})$, show that $\mathbb{Q}$ is neither a closed set nor an open set.

3. (a) Attempt any One of the following: (8)
   (i) State and prove Nested interval theorem in $\mathbb{R}$.
   (ii) Let (X,d) be a metric space and A be a subset of X. Show that $p \in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p.

   (b) Attempt any Two of the following: (12)
   (i) Let (X,d) be a metric space and (x_n) be a Cauchy sequence in X. If (x_n) has a convergent subsequence then prove that sequence (x_n) itself is convergent.
   (ii) Let (X,d) be a metric space and Y be a non-empty subset of X. Prove that a subset G of Y is open in the subspace (Y,d) if and only if $G = V \cap Y$ where V is an open set in (X,d).
   (iii) Prove that the metric space $(\mathbb{R}^2, d_1)$ is complete where the metric d_1 is given by $d_1((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.
   (iv) If $f : [a,b] \longrightarrow \mathbb{R}$ is a continuous such that f takes only rational values then show that f is a constant function.

4. (a) Attempt any One of the following: (8)
   - (i) Show that a compact subset of a metric space is closed and bounded. Give an example to show that a closed and bounded subset need not be compact.
   - (ii) Consider the metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that if A is closed and bounded then A satisfies Hein-Borel property.

   (b) Attempt any Two of the following: (12)
   - (i) Let A, B be compact subsets of a metric space (X, d). Show that $A \cup B$ and $A \cap B$ are compact subsets of (X, d).
   - (ii) Prove that a closed subset of a compact metric space is compact.
   - (iii) Consider the metric space $(\mathbb{R}, d)$, where d is the usual distance. Show that $\{\left(\frac{1}{n},\ 1\right) : n \in \mathbb{N}\}$ is an open cover of $(0, 1)$. Is $(0, 1)$ compact? Justify your answer.
   - (iv) Prove or disprove:
     - (I) Interior of a compact set is compact.
     - (II) A closed ball $B[x, r]$ in a metric space is compact.

5. Attempt any Four of the following: (20)

   (a) $\|\ \|_1$ and $\|\ \|_2$ are norms on $\mathbb{R}^2$ where for $x = (x_1, x_2) \in \mathbb{R}^2$, $\|x\|_1 = \sum_{i=1}^{2} |x_i|$,

   $\|x\|_2 = \sqrt{\sum_{i=1}^{2} x_i^2}$. Show that $\|x\|_2 \leq \|x\|_1$ and $\|x\|_1 \leq \sqrt{2}\ \|x\|_2$ for $x \in \mathbb{R}^2$.

   (b) Define distance of a point p from set A in a metric space (X, d). If $A \subseteq X$ then show that $|d(x, A) - d(y, A)| \leq d(x, y), \forall x, y \in X$.

   (c) Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X. If (x_n) is bounded in (X, d_1) then (x_n) is bounded in (X, d_2).

   (d) Prove that in a discrete metric space every Cauchy sequence is eventually constant. Hence deduce that a discrete metric space is complete.

   (e) Consider the set $K = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ in a metric space $(\mathbb{Q}, d)$ where d is a usual metric from $\mathbb{R}$. Is the set K compact in $(\mathbb{Q}, d)$? Justify your answer.

   (f) Prove that a subset of a discrete metric space is compact if and only if it is finite.

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