(3 Hours) **[Total Marks: 100]**

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose the correct alternative in each of the following: (20)

i. If D is the unit sphere $x^2 + y^2 + z^2 \leq 1$ then $\iiint_D zdV$ is equal to

(a) 0 (b) $\frac{2\pi}{3}$

(c) $\frac{4\pi}{3}$ (d) None of these

ii. The volume V of the solid above the region $R = \{(r,\theta)/1 \leq r \leq 3, 0 \leq \theta \leq {}^{\pi}/_{4}\}$ and under the surface $z = e^{x^2+y^2}$ is

(a) πe (b) $\pi e(e-1)$

(c) $\frac{\pi}{8}(e^9 - e)$ (d) $\frac{\pi}{8}e$

iii. If $f(x,y) = k$, k constant and $R = [a,b] \times [c,d]$ then $\iint_R kdA$ equals

(a) $k(b-a)(d-c)$ (b) $k(c-a)(d-b)$

(c) $k(b-a)(d-a)$ (d) data insufficient.

iv. A parameterization α of a circle of radius 2 centered at the origin in the X Z plane is given by

(a) $\propto: [0, 2\pi] \to \mathbb{R}^3,\ \propto(t) = (2\,cost\ , 2\sin t\,, 0\,)$

(b) $\propto: [0, 2\pi] \to \mathbb{R}^3,\ \propto(t) = (2\,cost\ , 2\sin t\,, 1\,)$

(c) $\propto: [\pi, 3\pi] \to \mathbb{R}^3,\ \propto(t) = (2\,cost\,, 0\,,\ 2\sin t)$

(d) $\propto: [0, 2\pi] \to \mathbb{R}^3,\ \propto(t) = (0\,, 2\,cost\ , 2\sin t)$

v. $I = \int_c 2ydx + 2xdy$ where C is the path $(t^9, sin^9(\pi t/2))$; $0 \leq t \leq 1$ Then I is

(a) $1/2$ (b) 2

(c) $\pi/2$ (d) None of these

vi. If $\oint_c P\,dx + Q\,dy = 0$ around every closed path C in a simply connected region R then

(a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ if P and Q are C^1 function

(b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ always

(c) $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$

(d) None of the above

vii. The surface area of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1) is

(a) $\sqrt{3}$ (b) $\frac{\sqrt{3}}{2}$

(c) $2\sqrt{3}$ (d) $1/2$

viii. The fundamental vector product for the cone
$x = r\cos\theta,\ y = r\sin\theta,\quad z = r,\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1$ is
(a) $(-r\cos\theta, -r\sin\theta,\ r)$ (b) $(r\cos\theta,\ r\sin\theta,\ r)$
(c) $(-r\cos\theta,\ r\sin\theta,\ r)$ (d) None of these

ix. $F(x,y,z) = (3xz,\ -5yz,\ z^2)$ and curl $(pyz^2, 0, qxyz) = F$. Then value of p and q are
(a) -1 & 3 (b) 1 & -3
(c) 1 & 3 (d) None of these

x. The surface integral $\iint_S\ ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}.dS$ over the surface of a unit sphere enclosing a volume V is
(a) $(a+b+c)\ 4\pi$ (b) $(a+b+c)V$
(c) $(a+b+c)4\pi^2$ (d) None of these

Q.2 a) Attempt any ONE. (08)

i. State and prove Fubini's Theorem for a rectangular domain in $\mathbb{R}^2$.

ii. If U is an open set in $\mathbb{R}^2$ containing the rectangle $[a,b] \times [c,d]$ and $f: U \to \mathbb{R}$ is continuously differentiable function then show that
$g'(x) = \int_c^d \frac{\partial f}{\partial x}(x,y)dy$ where $g(x) = \int_c^d f(x,y)dy,\ \forall x \in [a,b]$.

b) Attempt any TWO. (12)

i. If $S = \{(x,y): a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$ is a region in $\mathbb{R}^2$ where $\phi_1, \phi_2: [a,b] \to \mathbb{R}$ are continuous and a function $f: S \to \mathbb{R}$ is continuous in the interior of S with $f(x,y) \ge 0\ \forall (x,y) \in S$ then prove that $\iint_S\ f \ge 0$.

ii. Evaluate the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} (9-y^2)^{3/2} dydx$ by reversing the order of integration.

iii. Evaluate the integral $\iiint_S\ z\,dxdydz$ where S is the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 9$.

iv. Using cylindrical co-ordinates find the volume of the solid region S in $\mathbb{R}^3$ bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.

Q.3 a) Attempt any ONE. (08)

i. Suppose F is a continuous vector field defined on an open connected set U in $\mathbb{R}^n$. Define a function $\phi: U \to \mathbb{R}$ by $\phi(v) = \int_{v_0}^{v} F$ where v_0 is a fixed point in U and F is conservative. Show that $\nabla\phi(v) = F(v)\ \ \forall v \in U$.

ii. State and prove Green's Theorem for a rectangle. Further state Green's theorem for a closed region D in $\mathbb{R}^2$ whose boundary is a simple closed curve C. Show that area of region D $= \oint_C\ xdy$.

b) Attempt any TWO. (12)

i. Evaluate the line integral $\int_{(-1,2)}^{(3,1)} (y^2 + 2xy)dx + (x^2 + 2xy)\,dy$.

ii. Using Green's theorem evaluate the line integral $\oint_C 2x\cos y\,dx + x^2 \sin y\,dy$, where C is the positively oriented boundary of the region R enclosed between $y = x^2$ and $y = x$.

iii. Define the line integral of a vector field F defined on an open set U in $\mathbb{R}^n$ along an oriented curve Γ in U. If Γ and Γ' are two equivalent but orientation reversing curves in U, show that $\int_\Gamma F = -\int_{\Gamma'} F$.

iv. Find the work done by the force $F = (-4xy, 8y, z)$ as the point of application moves along the curve of intersection of the parabolic cylinder $y = x^2$ and the plane $z = 1$ from (0,0,1) to (2,4,1).

Q.4 a) Attempt any ONE. (08)

i. Let $S = \bar{r}(T)$ be a smooth parametric surface described by a differentiable function $\bar{r}$ defined on region T. Let f be defined and bounded on S. Define surface integral of f over S. If R and r are smoothly equivalent functions, $R(s,t) = \bar{r}(G(s,t))$ where $G(s,t) = u(s,t)\hat{\imath} + v(s,t)\hat{\jmath}$ being continuously differentiable. Then show that $\iint_{r(A)} f dS = \iint_{R(B)} f\,d\,S$ where $G(B) = A$.

ii. State and prove Stokes' Theorem for an oriented smooth, simple parameterized surface in $\mathbb{R}^3$ bounded by a simple, closed curve traversed counter clockwise assuming general form of Green's Theorem.

b) Attempt any TWO. (12)

i. If S and C satisfy hypothesis of Stokes' Theorem and f, g have continuous second order partial derivative, prove with usual notations

(a) $\int_C (f\nabla g).dr = \iint_S (\nabla f \; X \; \nabla g)\,\hat{n}$ ds

(b) $\int_C (f\nabla f).dr = 0$

(c) $\int_C (f\nabla g + g\nabla \mathrm{f}).dr = 0$

ii. Evaluate surface integral of $f(x,y,z) = x^2 + y^2$ where S is the surface of the paraboloid $x^2 + y^2 = 4 - z$ above the XY-plane.

iii. Use Stokes' theorem to find $\iint_S$ (curl F) · n dS where $\mathrm{F(x,y,z) = (y,z,x)}$ and S is the surface of the paraboloid $\mathrm{z = 1 - x^2 - y^2; z \geq 0}$.

iv. Evaluate $\iint_S f(x,y,z).\hat{n}ds$ where $f(x,y,z) = (x,y,z)$ and S is the surface of the cylinder $x^2 + y^2 = 4$ between $0 \leq z \leq 4$.

Q.5 Attempt any FOUR. (20)

a) Evaluate $\iiint_S dV$ where region S is bounded by the three co-ordinate planes and the plane $x + y + z = 1$.

b) Evaluate $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{dxdy}{1+x^2+y^2}$ by converting into polar coordinates.

c) Evaluate the line integral of $f(x,y,z) = x + y + z$, along the path $\gamma(t) = (\sin t, \cos t, t)$, $0 \leq t \leq 2\pi$.

d) Find a potential function of F where $F(x,y,z) = (e^x \sin z + 2yz, 2xz + 2y, \quad e^x \cos z + 2xy + 3z^2)$.

e) Find surface area of S where S is the surface of the sphere $x^2 + y^2 + z^2 = 16$ in first octant.

f) Use Gauss Divergence theorem to find $\iint_S F \cdot \bar{n} dS$: where $F(x,y,z) = (y - x, z - y, y - x)$ and S is the cube bounded by the planes $x = \pm 1, y = \pm 1, z = \pm 1$.
