3 Hours] **[Total Marks: 100**

N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

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1. Fill in the blank by choosing the correct option.
   i. Consider $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + 2y + z = 0, 3x + 3y - 2z = 0, x + y - 3z = 0\}$. Then $dim\mathbb{R}^3/W$ is ——— (2)
   (a) 2 (b) 3 (c) 1 (d) None of these.
   ii. $\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$ is ——— (2)
   (a) an orthogonal matrix of reflection
   (b) an orthogonal matrix of rotation
   (c) not an orthogonal matrix
   (d) None of these.
   iii. If $p_1(t) = t^2 + a_1t + a_0$ is characteristic polynomial of A and $p_2(t) = t^2 + b_1t + b_0$ is characteristic polynomial of A^2 then $b_0 =$——— (2)
   (a) a_0 (b) a_0^2 (c) $\frac{1}{a_0}$ (d) None of these.
   iv. 0 is an eigen value of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ if and only if ———-. (2)
   (a) T is invertible (b) T is not invertible
   (c) $T = 0$ (d) None of these.
   v. Let $A = [a_{ij}]$ be a 10×10 matrix with $aij = \begin{cases} 1 & \text{if } i + j = 11 \\ 0 & \text{otherwise} \end{cases}$. (2)
   Then the set of eigen values of A is ———.
   (a) $\{0, 1\}$ (b) $\{-1, 1\}$
   (c) $\{0, 11\}$ (d) None of these.
   vi. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal transformation of reflection in a straight line passing through origin then T has ——— eigen values. (2)
   (a) only two (b) only one (c) No (d) None of these.
   vii. The minimal polynomial of $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ for any $\alpha \neq 0$ is ———. (2)
   (a) $x^2 - \alpha$ (b) $x^2 - x$ (c) $(x - 1)^2$ (d) None of these.

[Turn over]

viii. Let A and B be 3×3 non-diagonal matrices over $\mathbb{R}$ such that $A^2 = A, B^2 = -I$ then ———— (2)
(a) A is diagonalisable. (b) B is diagonalisable.
(c) A and B are diagonalisable. (d) None of these.

ix. $A = \begin{pmatrix} 1 & 7 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{pmatrix}$ is ————. (2)
(a) diagonalisable but not orthogonally diagonalisable.
(b) orthogonally diagonalisable
(c) not diagonalisable
(d) None of these.

x. The rank and signature of the quadratic from $Q(x) = -3x_1^2+5x_2^2+2x_1x_2$ are ————- (2)
(a) 2 and 2 (b) 2 and 0
(c) 2 and -2 (d) None of these.

2. (a) Answer any **ONE**
i. Let V be an n dimensional inner product space and W be a subspace of V of dimension $n-1$. Let u be a unit vector orthogonal to W. Show that $T : V \to V$ defined by $T(x) = x - 2\langle x, u\rangle u$ is an orthogonal linear transformation such that $T(w) = w, \ \forall \ w \in W$ and $T(u) = -u$. (8)
ii. State and prove the Cayley Hamilton theorem. (8)

(b) Answer any **TWO**
i. Let V be a finite dimensional inner product space and $f : V \to V$ be an isometry, then show that there exists unique $x_0 \in V$ and an unique orthogonal linear transformation $T : V \to V$ such that $f = L_{x_0} \circ T$ where $L_{x_0} : V \to V$ is a translation map defined as $L_{x_0}(X) = X + X_0$. (6)
ii. Let V be a finite dimensional inner product spaace and $T : V \to V$ be a linear transformation. Prove that T is orthogonal if and only if $\|T(X)\| = \|X\| \quad \forall \ X \in V$. (6)
iii. If $A_{2\times 2}$ matrix has the characteristic polynomial $x^2 + 2x - 1$, then find the value of det $(2I_2 + A)$. (6)
iv. If u is a unit column vector in $\mathbb{R}^n$ and $A = I - 2uu^t$. Then prove that A is an orthogonal matrix. (6)

3. (a) Answer any **ONE**

i. If λ is an eigen value of a real $n \times n$ matrix A, then (8)

(p) λ is an eigen value of A^t.

(q) λ^k is an eigen value of A^k for $k \in \mathbb{N}$. Hence $f(\lambda)$ is an eigen value of $f(A)$, for a polynomial $f(x)$ over $\mathbb{R}$.

(r) If A is invertible, then λ^{-1} is an eigen value of A^{-1}.

ii. Define the minimal polynomial of a square matrix A. Prove that similar matrices have same minimal polynomials. Is the converse true? Justify. (8)

(b) Answer any **TWO**

i. Let $A_{n\times n}$ be a real matrix. Show that eigen vectors corresponding to distinct eigen values, $\lambda_1, \lambda_2, \cdots, \lambda_k$, of A are linearly independent. (6)

ii. Define invariant subspace. Let V be a finite dimension vector space and $T : V \to V$ be a linear transformation. Show that eigen space of T is invariant under T. (6)

iii. Find the characteristic polynomial and the minimal polynomial of $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. (6)

iv. Let A be a 13×13 real matrix of rank 1 and $P(t)$ be the characteristic polynomial of A then prove that $P(t) = t^{12}(t - \text{trace}\,(A))$. (6)

4. (a) Answer any **ONE**

i. Show that real symmetric matrix of order n is orthogonally diagonalizable. (8)

ii. Define algebraic and geometric multiplicities of an eigen value of a square matrix. Show that the geometric multiplicity of an eigen value does not exceeds its algebraic multiplicity. (8)

(b) Answer any **TWO**

i. Show that a real $n \times n$ is diagonalisable if and only if there is basis of $\mathbb{R}^n$ consisting of eigen vectors of A. (6)

ii. Let A be an $n \times n$ real symmetric matrix. Then show that $\langle AX, X\rangle > 0$ for all non-zero $X \in \mathbb{R}^n$ if and only if each eigenvalue of A is positive. (6)

[Turn over]

iii. Determine constants a, b, c so that $\begin{pmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{pmatrix}$ is diagonalisable (6)

iv. Let $A_{3\times 3}$ real matrix having 1, -1 3 as eigen values. Determine which of the matrices in S are non-singular where $S = \{A^2 + A, A^2 - A, A^2 + 3A, A^2 - 3A\}$. Justify your answer. (6)

5. Answer any **FOUR**

(a) Prove or disprove: If the characteristic polynomial of a matrix $A_{n\times n}$ is same as minimal polynomial then A has distinct eigen values. (5)

(b) Find an orthogonal transformation in $\mathbb{R}^3$ which represents reflection with respect to the plane $x - 2y + z = 0$. (5)

(c) Find the eigenvalues and the bases of the corresponding eigen spaces for $\begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$. (5)

(d) Let λ_1, λ_2 be the distinct eigenvalues of A with X_1, X_2 as corresponding eigen vectors, then show that $X_1 + X_2$ is not an eigen vector of A. (5)

(e) Find the condition on k so that $3x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_3 + 2kx_2x_3$ is positive definite by stating the necessary result. (5)

(f) Identify the conic $5x^2 + 4xy + 5y^2 - 9 = 0$. (5)

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