N.B.: 1) Question No.1 is compulsory.

2) Attempt any three from the remaining questions.

3) Assume suitable data if necessary.

1. (a) Determine the constants a, b, c, d if f(z) = x^2 + 2axy + by^2 + i(dx^2 + 2cxy + y^2)
   is analytic.

(b) Find a cosine series of period 2\pi to represent \sin x in 0 \leq x \leq \pi

(c) Evaluate by using Laplace Transformation \( \int_0^\infty e^{-3t} \cos t \, dt \).

(d) A vector field is given by \( \mathbf{F} = (x^2+xy^2) \mathbf{i} + (y^2+x^2) \mathbf{j} \). Show that \( \mathbf{F} \) is
   irrotational and find its scalar potential. Such that \( \mathbf{F} = \nabla \varphi \).

2. (a) Solve by using Laplace Transform
   
   \[(D^2 + 2D + 5) y = e^{-t} \sin t, \text{ when } y(0) = 0, \ y'(0) = 1.\]

(b) Find the total work done in moving a particle in the force field
   \( \mathbf{F} = 3xy \mathbf{i} - 5z \mathbf{j} + 10x \mathbf{k} \) along \( x=t^2+1, y=2t^3, z=t^3 \) from \( t=1 \) and \( t=2 \).

(c) Find the Fourier series of the function \( f(x) = e^x, \quad 0 < x < 2\pi \) and
   \[ f(x + 2\pi) = f(x). \] Hence deduce that the value of \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} \).

3. (a) Prove that \( J_{1/2}(x) = \frac{2}{\sqrt{\pi x}} \sin x \)

(b) Verify Green's theorem in the plane for \( \int \left( x^2 - y \right) \, dx + (2y^2 + x) \, dy \)
   around the boundary of region defined by \( y = x^2 \) and \( y = 4 \).

(c) Find the Laplace transforms of the following.
   
   i) \( e^t \int_0^t \frac{\sin u}{u} \, du \) \hspace{1cm} ii) \( t \sqrt{1 + \sin t} \)

4 (a) If \( f(x) = C_1 Q_1(x) + C_2 Q_2(x) + C_3 Q_3(x) \), where \( C_1, C_2, C_3 \) are constants and \( Q_1, Q_2, Q_3 \) are orthonormal sets on \((a,b)\), show that \[
\int_a^b [f(x)]^2 \, dx = c_1^2 + c_2^2 + c_3^2.
\]

(b) If \( v = e^x \sin y \), prove that \( v \) is a Harmonic function. Also find the corresponding harmonic conjugate function and analytic function.

(c) Find inverse Laplace transforms of the following.

i) \( \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \)

ii) \( \frac{s+2}{s^2 - 4s + 13} \)

5 (a) Find the Fourier series if \( f(x) = |x|, \quad -k < x < k \)

Hence deduce that \( \sum \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \).

(b) Define solenoidal vector. Hence prove that \( \mathbf{F} = \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \) is a solenoidal vector.

(c) Find the bilinear transformation under which \( 1, i, -1 \) from the \( z \)-plane are mapped onto \( 0, 1, \infty \) of \( w \)-plane. Further show that under this transformation the unit circle in \( w \)-plane is mapped onto a straight line in the \( z \)-plane. Write the name of this line.

6 (a) Using Gauss's Divergence Theorem evaluate \( \iint_S \mathbf{F} \cdot d\mathbf{S} \) where \( \mathbf{F} = 2xy \mathbf{i} - y^2 \mathbf{j} + 4xz^2 \mathbf{k} \) and \( S \) is the region bounded by \( y^2 + z^2 = 9 \) and \( x = 2 \) in the first octant.

(b) Define bilinear transformation. And prove that in a general, a bilinear transformation maps a circle into a circle.

(c) Prove that \( \int x \frac{1}{2/3} (x^{3/2}) \, dx = -\frac{2}{3} x^{1/2} \frac{1}{1/3} (x^{3/2}) \).