Q.P. 37729

MATHEMATICS SOLUTION

(NOV 2018 SEM 4 MECHANICAL)

Q1(a) State that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is non derogatory. **(5M)**

Solution:

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0$$

$(1 - \lambda) [(3 - \lambda) (5 - \lambda) - 16] - 2[2 (5 - \lambda) - 12] + 3[8 - 3 (3 - \lambda)] = 0$

$(1 - \lambda) [(-1 - 8\lambda + \lambda^2] - 2[- 2 - 2\lambda] + 3[- 1 + 3\lambda)] = 0$

$\lambda^3 - 9\lambda^2 - 6\lambda = 0$

$\lambda [\lambda^2 - 9\lambda - 6] = 0$

Since, all roots are distinct and since the characteristic equation is satisfied by A. The degree of minimal equation is equal to 3 and hence A is non-derogatory.

(b) Determine all basic solutions to the following problem

Maximise $z = x_1 + 3x_2 + 3x_3$

Subjected to: $1x_1 + 2x_2 + 3x_3 = 4$,

$2x_1 + 3x_2 + 5x_3 = 7$,

$x_1, x_2, x_3 \geq 0$ **(5M)**

Solution:

No. of basic solution	Non-basic variables =0	Basic variables	Equations and the values of the basic variables	Is the solution feasible?	Is the solution degenerate?	Value of z	Is the solution optimal?
1.	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	Yes	No	5	Yes
2.	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	Yes	No	4	No
3.	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $x_2 = -1, x_3 = 2$	No	No	---	---

(c) Prove that $\bar{F} = (2xy + z)i + (x^2 + 2yz^3)j + (3y^2z^2 + x)k$ is an irrotational vector and find the corresponding scalar ϕ such that $\bar{F} = \nabla\phi$. **(5M)**

Solution:

$$\text{Curl}(\bar{F}) = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xy + z & x^2 + 2yz^3 & 3y^2z^2 + x \end{vmatrix}$$

$$= (6yz^2 - 6yz^2)i + (1 - 1)j + (2x - 2x)k$$

$$= 0$$

$\bar{F}$ is irrotatonal.

Since $\bar{F}$ is irrotatonal there exists a scalar function ϕ, such that $\bar{F} = \nabla.\phi$

$$(2xy + z)i + (x^2 + 2yz^3)j + (3y^2z^2 + x)k = \frac{\delta\phi}{\delta x}i + \frac{\delta\phi}{\delta y}j + \frac{\delta\phi}{\delta z}j$$

$$\frac{\delta\phi}{\delta x} = 2xy + z \; ; \; \frac{\delta\phi}{\delta y} = (x^2 + 2yz^3) \; ; \frac{\delta\phi}{\delta z} = (3y^2z^2 + x)$$

$$d\phi = \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz$$

$$= (2xy + z)dx + (x^2 + 2yz^3)dy + (3y^2z^2 + x)dz$$

$$= 2xy.dx + z.dx + x^2.dy + 2yz^3.dy + 3y^2z^2.dz + x.dz$$

$$= d(x^2y + zx + x^2y + \frac{y^2z^4}{2} + y^2z^3 + xz)$$

$$= d\left(2x^2y + 2zx + y^2\left(\frac{z^4}{2} + z^3\right)\right)$$

$$\Phi = 2x^2y + 2zx + y^2\left(\frac{z^4}{2} + z^3\right) + c$$

Where, c is constant of integration

(d) Can it be concluded that the average life span of an Indian is more than 71 years, if a random sample of 900 Indians has an average life span 72.8 years with standard deviation of 7.2 years? **(5M)**

Solution:

Null Hypothesis H_o: μ = 70 years

Alternate Hypothesis H_a: μ ≠ 70 years

Test statistic: $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

Since we are given standard deviation of the sample, we put

$\bar{X} = 71.8$, $\mu = 70$, $\sigma = 7.2$, $n = 100$

$$Z = \frac{71.8 - 70}{7.2/\sqrt{100}} = 2.5$$

Level of significance: α = 0.05

Critical value: the value of z_α at 5% level of significance is 1.96

Decision: Since the computed value $|Z| = 2.5$ is greater than the critical value z_α = 1.96, the null hypothesis is rejected.

Q2 (a) Show that the matrix A=$\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalizable and hence find the transforming matrix and diagonal matrix. **(6M)**

Solution:

The characteristic equation of A is

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-3) = 0$$

λ = 1, 2, 3.

Since, all eigen values are distinct the matrix A is diagonalize.

(i) For λ = 1, $[A - \lambda_1 I] = 0$

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$7x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 4x_2 - 2x_3 = 0$

By Cramer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{x_2}{6} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2} = t$$

$x_1 = 4t;\ x_2 = 3t; x_1 = 2t$

$$X_1 = \begin{bmatrix} 4t \\ 3t \\ 2t \end{bmatrix} = t\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \qquad X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

Corresponding to eigenvalue 1, the eigenvector is $[4, 3, 2]'$.

(ii) For λ = 2, $[A - \lambda_2 I] = 0$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$6x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 5x_2 - 2x_3 = 0$

By Cramer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{x_2}{4} = \frac{x_3}{2}$$

$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = t$

$x_1 = 3t;\ x_2 = 2t; x_1 = t$

$X_2 = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $\qquad X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Corresponding to eigenvalue 2, the eigenvector is $[3, 2, 1]'$.

(iii) For λ = 3, [A $-\lambda_3 I$] = 0

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 8x_2 - 2x_3 = 0$

$4x_1 - 6x_2 - 2x_3 = 0$

By Cramer's Rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2}$

$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1} = t$

$x_1 = 2t;\ x_2 = t; x_1 = t$

$X_3 = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\qquad X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Corresponding to eigenvalue 3, the eigenvector is $[2, 1, 1]'$.

M = $[X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Since $M^{-1}AM = D$, the matrix A = $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ will be diagonalized to the diagonal matrix

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by the transforming matrix M = $\begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

(b)Verify Green's theorem for $\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the closed curve given by y □x^2; $y = \sqrt{x}$ **(6M)**

Solution:

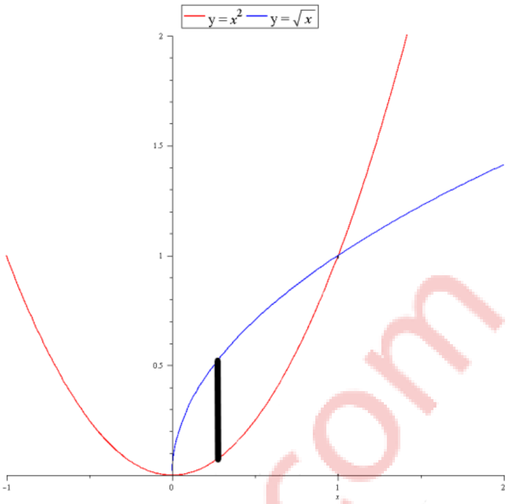
By Green's Theorem

$$\int_c P.dx + Q.dy = \iint_R \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y} .dx.dy$$

Here, P = $(3x^2 - 8y^2)$; Q= $(4y - 6xy)$

$\frac{\delta Q}{\delta x} = -6y\,;\ \frac{\delta P}{\delta y} = -16y$

Along, y=x^2 and dy=2x.dx and x varies from (0,1)



$$\int_c P.dx + Q.dy$$

$$= \int_0^1 (3x^2 - 8y^2).dx + (4y - 6xy).\text{dy}$$

$$= \int_0^1 (3x^2 - 8x^4).dx + 2x(4x^2 - 6x^3).dx$$

$$= \left(\frac{3x^3}{3} - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5}\middle| x = 0\ to\ 1\right)$$

$$= 1 - \frac{8}{5} + \frac{8}{4} - \frac{12}{5} = -1$$

Along $y = \sqrt{x},\ dy = \frac{1}{2\sqrt{x}}dx$

$$\int_c P.dx + Q.dy$$

$$= \int_0^1 (3x^2 - 8x).dx$$

$$+ (4\sqrt{x} - 6x\sqrt{x}).\frac{1}{2\sqrt{x}}.dx$$

$$= \int_0^1 (3x^2 - 8x).dx + (2 - 3x).dx$$

$$= \left(\frac{3x^3}{3} - \frac{8x^2}{2} + 2x - \frac{3x^2}{2}\middle| x = 0\ to\ 1\right)$$

$$= 1 - 4 + 2 - \frac{3}{2} = \frac{-5}{2}$$

$$\iint_R \frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}.dx.dy = \int_0^1 \int_{x^2}^{\sqrt{x}} (-22y).dx.dy$$

$$= \int_0^1 \left(-11y^2 \middle| y = x^2\ to\ \sqrt{x}\right).dx$$

$$= \int_0^1 -11x - (-11x^4).dx$$

$$= \int_0^1 -11x + 11x^4.dx$$

$$= \left(\frac{-11x^2}{2} + \frac{11x^5}{5}\middle| x = 0\ to\ 1\right)$$

$$= \frac{-33}{10}$$

(c)Solve the following problem by simplex method

Maximize $z = 3x_1 + 2x_2$

$3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_2 \leq 6$

$x_1, x_2 \geq 0$ **(8M)**

Solution:

We first express the problem in standard form

$z - 3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

$3x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 18$

$x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 = 4$

$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 7$

We now express the above information in tabular form

Iteration Number	Basic Variables	Coefficient of					R.H.S Solution	Ratio
		x_1	x_2	s_1	s_2	s_3		
0	Z	-3	-2	0	0	0		
s_2 leaves	s_1	3*	2	1	0	0	18	6
x_1 enters	s_2	1	0	0	1	0	4	4
	s_3	0	1	0	0	1	6	---
1	Z	0	-2	0	3	0	12	
s_1 leaves	s_1	0	2*	1	-3	0	6	3
x_2 leaves	x_1	1	0	0	1	0	4	---
	s_3	0	1	0	0	1	6	8
2	Z	0	0	1	0	0	18	
	x_2	0	1	1/2	-3/2	0	3	
	x_1	1	0	0	1	0	4	
	s_3	0	0	-1/2	3/2	1	3	

$x_1 = 4$; $x_2 = 3$; $z_{max} = 18$

Q3 (a) Use Stoke's theorem evaluate $\int \bar{F}.d\bar{r}$. $\bar{F} = 2y(1-x)i + (x - x^2 - y^2)j + (x^2 + y^2 + z^2)k$ where s is the surface of the plane x + y + z = 2 which is the first octant. **(6M)**

Solution:

By Stokes theorem $\int_c \bar{F}d\bar{r} = \iint_s \bar{N}.\nabla.\bar{F}ds$

Now, $\nabla X \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2y(1-x) & (x - x^2 - y^2) & (x^2 + y^2 + z^2) \end{vmatrix}$

$= (2y - 0)i - (2x - 0)j + (1 - 2x - 2 + 2x)k$

$= 2yi + 2xj - k$

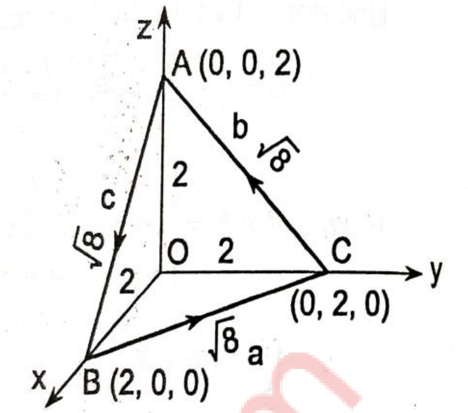
Further $\phi = x + y + z - 2$

Normal to the plane ABC,

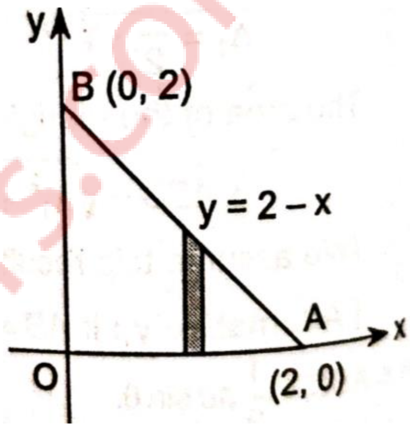
$\nabla\phi = \frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k = i + j + k$

Unit normal to the plane ΔABC

$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{i+j+k}{\sqrt{3}}$



Lets ds be the element in the plane of the plane ABC. Then its projection in xy-plane will be dx dy. If the θ is the angle between plane ABC and the xy-plane, then



$dx.dy = cos\theta.ds$ and $\cos\theta = \bar{N}.k = \left(\frac{i+j+k}{\sqrt{3}}\right).k = \frac{1}{\sqrt{3}}$

$dx.dy = \frac{1}{\sqrt{3}}.ds \qquad ds = \sqrt{3}.dxy$

$\bar{N}.\nabla.\bar{F}ds = \frac{i+j+k}{\sqrt{3}}.(2yi - 2xj - k)\sqrt{3}.dx.dy = (2y - 2x - 1).dx.dy$

$\iint_s \bar{N}.\nabla.\bar{F}ds = \iint_{\Delta OAB}(2y - 2x - 1).dx.dy$

$\int_{x=0}^{2}\int_{y=0}^{2-x}(2y - 2x - 1).dx.dy = \int_0^2(y^2 - 2xy - y|y = 0\ to\ 2 - x)$

$= \int_0^2[(2 - x)^2 - 2x(2 - x) - (2 - x)].dx$

$= \int_0^2[(2 - x)^2 - 2(2x - x^2) - (2 - x)].dx$

$= \left(\frac{-(2-x)^3}{3} - 2\left(x^2 - \frac{x^3}{3}\right) + \frac{(2-x)^2}{2}\middle|x = 0\ to\ 2\right)$

$= \left(-2\left(4 - \frac{8}{3}\right) - \left(-\frac{8}{3} + 2\right)\right) = -2$

(b)The standard deviations calculated from two random samples of sizes 9 and 13 are 1.99 and 1.9. Can the samples be regard as drawn from the normal populations with same standard deviations? (Given: $F_{0.025}$ = 3.51 with d.o.f. 8 & 12 and $F_{0.025}$ = 4.20 with d.o.f. 12 & 8 or $F_{0.005}$ = 4.50 with d.o.f. 8 & 12) (6M)

Solution:

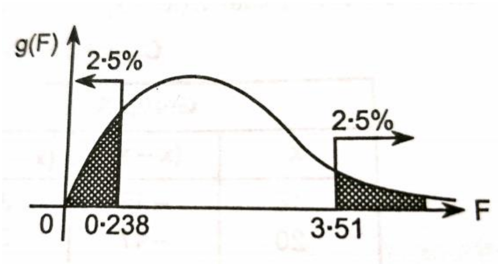
Null Hypothesis Ho: $\sigma_1^2 = \sigma_1^2$

Alternative Hypothesis Ha: $\sigma_1^2 \neq \sigma_1^2$

Calculations of Test Statistic: $F = \frac{n_1 s_1^2/(n_1-1)}{n_2 s_2^2/(n_2-1)}$

We are given n_1=9, n_2=13, $s_1^2 = 1.99^2, s_2^2 = 1.9^2$

$F = \frac{9*1.99^2/(9-1)}{13*1.9^2/(13-1)} = \frac{4.455}{3.91} = 1.139$



Level of significance $\alpha = 0.05$

Degree of freedom $v_1 = n_1 - 1 = 8$ for the numerator
$v_2 = n_2 - 1 = 12$ for the denominator

Critical Value: The table value
$F_{(8,12)}(0.025) = 3.51$
$F_{(12,8)}(0.025) = 4.20$

$\frac{1}{F(12,8)(0.025)} = \frac{1}{4.20} = 0.238$

Decision: Since the calculated value F=1.139 lies between 0.238 and 3.51, we accept the null hypothesis.

(c) Use Penalty Method (Big M method) to solve the following L.P.P.

Minimise $z = 6x_1 + 4x_2$

Subjected to: $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$ **(8M)**

Solution:

We introduce slack variable s_1, s_2 in the first two constraints, surplus variable s_3 and the artificial variable A_3 in the third constraint, and big penalty in the object function.

Maximise $z = 6x_1 + 4x_2 - 0s_1 - 0s_2 - 0s_3 - MA_3$

Subjected to: $2x_1 + 3x_2 + s_1 + 0s_2 + 0s_3 + 0A_3 = 30$

$3x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 + 0A_3 = 24$

$x_1 + x_2 + 0s_1 + 0s_2 + s_3 + A_3 = 3$

We now eliminate the term $-MA_3$ from the object function by adding M times the third constraint to the object function.

$z = 6x_1 + Mx_1 + 4x_2 + Mx_2 - 0s_1 - 0s_2 - Ms_3 - 0A_3 - 3M$

$z - (6 + M)\,x_1 - (4 + M)\,x_2 - 0s_1 - 0s_2 - Ms_3 - 0A_3 = -3M$

Iteration No.	Basic Var.	coefficient of						R.H.S. Solution	Ratio
		x_1	x_2	s_1	s_2	s_3	A_3		
0	z	-6-M	-4-M	0	0	M	0	-3M	
A_3 leaves	s_1	2	3	1	0	0	0	30	30/2 = 15
x_1 enters	s_2	3	2	0	1	0	0	24	24/3 = 8
	A_3	1*	1	0	0	-1	1	3	3/1 = 3
1	z	0	2	0	0	-6		18	
s_2 leaves	s_1	0	1	1	0	2		24	24/2 = 12
s_3 enters	s_2	0	-1	0	1	3*		15	15/3 = 5
	x_1	1	1	0	0	-1		3	3/-1 = -3
2	z	0	0	0	2	0		48	
s_1 leaves	s_1	0	5/3*	1	-2/3	0		14	14/(5/3) = 42/5
x_2 enters	s_2	0	-1/3	0	1/3	1		5	1*-3 = …
	x_1	1	2/3	0	1/3	0		8	8/(3/2) = 12
3	z	0	0	0	2	0		48	
	x_2	0	1	3/5	-2/5	0		42/5	
	s_2	0	0	1/5	1/5	1		39/5	
	x_1	1	0	-2/5	3/5	0		12/5	

$x_1 = 12/5$, $x_2 = 42/5$, $z_{max} = 48$.

Q4(a) Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Hence find A^{-1}. **(6M)**

Solution:

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$(2 – \lambda) [(1 – \lambda) (2 – \lambda) – 0] – 1[0 – 0(2 – \lambda)] + 1[1(1 – \lambda) – 0] = 0$

$(4 – 4\lambda + \lambda^2) (1 – \lambda) – (1 – \lambda) = 0$

$\lambda^3 – 5\lambda^2 – 7\lambda – 3 = 0$

This equation is satisfied by A.

In terms of the matrix A this means $A^3 – 5A^2 – 7A – 3I$ Now,

Multiply by A^{-1}, we get $A^2 – 5A – 7I – 3A^{-1}$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}=\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$A^2 - 5A - 7I - 3A^{-1} = 0$

$$A^{-1} = \frac{1}{3}\left\{-\begin{bmatrix}5 & 4 & 4\\0 & 1 & 0\\4 & 4 & 5\end{bmatrix} + 5\begin{bmatrix}2 & 1 & 1\\0 & 1 & 0\\1 & 1 & 2\end{bmatrix} + 7\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right\}$$

$$= \frac{1}{3}\begin{bmatrix}12 & 1 & 1\\0 & 11 & 0\\1 & 1 & 12\end{bmatrix}$$

(b) Marks obtained by students in an examination follow normal distributions. If 30% of students got below 35 marks and 10% got above 60 marks. Find mean and standard deviation. (6M)

Solution:

Let m and n be the mean and standard deviation of the distribution.

Since 30% students are below 35, and 20% are in between 35 and m.

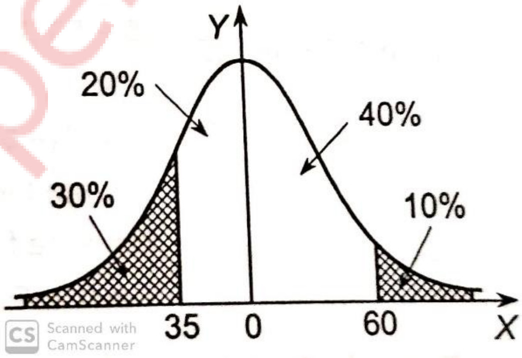
Since 10% students are above 40, and 40% are in between m and 60.

From the table we find that,

0.2 area corresponds to Z = 0.525

And 0.4 area corresponds to Z = 1.283

But 0.2 area is to the left of m hence Z = - 0.525



$\frac{35-m}{\sigma} = -0.525$; $\frac{60-m}{\sigma} = 1.283$………(i)

$35 - m = -0.525\sigma$

$60 - m = 1.283\sigma$

By subtracting, we get

$25 = 1.808\ \sigma$

$\sigma = \frac{25}{1.808} = 13.83$

Putting this value of σ in (i), we get

$35 - m = -0.525 \times 13.83$

$m = 35 + 0.525 \times 13.83 = 42.26$

Hence, the mean = 13.83 and the standard deviation, σ = 42.26.

(c) Use the dual simplex method to solve the following L.P.P.

Minimize $z = x_1 + x_2$

Subjected to: $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq -1$

$x_1, x_2 \geq 0$ **(8M)**

Solution:

Minimise $z = x_1 + x_2$

Subject to $-2x_1 - x_2 \leq -2$

$x_1 + x_2 \leq -1$

Introducing the slack variables s_1, s_2, we have

Minimise $z = x_1 + x_2 - 0s_1 - 0s_2$

i.e. $z - x_1 - x_2 + 0s_1 + 0s_2$

Subject to $-2x_1 - x_2 + s_1 + 0s_2 = -2$

$x_1 + x_2 + 0s_1 + s_2 = -1$

Iteration Number	Basic Variable	Coeficient	Of			R.H.S. Solution
		x_1	x_2	s_1	s_2	
0	z	-1	-1	0	0	0
s_1 leaves	s_1	-2*	-1	1	0	-2
x_1 enters	s_2	1	1	0	1	-1
Ratio		1/2	1			
1	Z	0	-1/2	-1/2	0	1
	x_1	1	1/2	-1/2	0	1
	s_2	0	1/2	1/2	1	-2
Ratio		---	-1	-1	---	

Now s_2 row is negative, s_2 leaves. But since all ratios are negative, the L.P.P. has no feasible solution.

Q5 (a)Find e^A and 4^A. If A= $\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ **(6M)**

Solution:

The characteristic equation of A is

$$\begin{vmatrix} \frac{3}{2} - \lambda & 1/2 \\ 1/2 & \frac{3}{2} - \lambda \end{vmatrix} = 0$$

$(3/2-\lambda)^2 - ¼ = 0$

$9/4 - 3\lambda + \lambda^2 - ¼ = 0$

λ^2- 3λ +2=0

(λ-1)*(λ-2)=0

λ= 1, 2

1. For λ=1, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $2R_2$ and $2R_1$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By R_2 - R_1 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1+x_2 = 0$

putting x_2=-t , we get x_1=t

X_1= $\begin{bmatrix} t \\ -t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence the eigen values are 1, -1.

2. For λ=2, $[A - \lambda I]X = 0$ gives

$$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $2R_2$ and $2R_1$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By R_2-R_1 $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1+x_2 = 0$

$x_1=x_2$

putting x_2=t, we get x_1=t

X_2= $\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence the eigen values are 1, 1.

M = $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad\quad$ $|M| = 2$

$M^{-1} = \frac{adj\ A^{-1}}{|M|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Now, D=$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

If f(A)= e^A, $\quad$ f(D)=e^D=$\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$

If f(A)= 4^A, $\quad$ f(D)=4^D=$\begin{bmatrix} 4^1 & 0 \\ 0 & 4^2 \end{bmatrix}$

e^A = M f(D) M^{-1}

$$=\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}e^1 & 0\\0 & e^2\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}e+e^2 & -e+e^2\\-e+e^2 & e+e^2\end{bmatrix}$$

Similarly, replacing e by 4, we get

$$4^A=\frac{1}{2}\begin{bmatrix}20 & 12\\12 & 20\end{bmatrix}=\begin{bmatrix}10 & 6\\6 & 10\end{bmatrix}.$$

(b) A random discrete variable x has the probability density function given

X	**-2**	**-1**	**0**	**1**	**2**	**3**
P(X)	**0.1**	**k**	**0.2**	**2k**	**0.3**	**k**

Find k, the mean and the variance. **(6M)**

Solution:

Since $\sum p(X) = 1$

0.1 + k + 0.2 + 2k + 0.3 + k = 1

0.6 + 4k = 1

4k = 0.4

k = 0.1

Therefore, the probability distribution of X is

X	-2	-1	0	1	2	3
P(X)	0.1	0.1	0.2	0.2	0.3	0.1

E(X) = mean = $\sum p_i x_i$

$= (-2) * 0.1 + (-1) * 0.1 + 0 * 0.2 + 1 * 0.2 + 2 * 0.3 + 3 * 0.1$

$= 1.5$

E(X^2) = $\sum p_i {x_i}^2$

$= (-2) * 0.1^2 + (-1) * 0.1^2 + 0 * 0.2^2 + 1 * 0.2^2 + 2 * 0.3^2 + 3 * 0.1^2$

$= 0.22$

Variance = E(X) - E(X^2)

= 1.5 – 0.22

= 1.28

(c) In an experiment on immunizations of cattle from Tuberculosis the following results were obtained. Using $x^2 - test$ to determine the efficiency of vaccine in preventing tuberculosis.

	Affected	**Not affected**	**Total**
Inoculated	**290**	**110**	**400**
Not Inoculated	**310**	**90**	**400**
Total	**600**	**200**	**800**

Solution: **(8M)**

(i)Null Hypothesis H_o: There is no association between the vaccine and the not affected people.

Alternative Hypothesis H_a: There is association

(ii)On the basis of this hypothesis, the number in the first cell $= \frac{A \, X \, B}{N}$

Where, A = number of inoculated

B = number who are affected

N = Total number of vaccine

Expected frequency $= \frac{400 \, X \, 600}{800} = 300$

This is the frequency in the first cell.

The frequencies in the remaining cells are 400-300=100, 600-300=300, 400-300=100.

Calculation of $\boldsymbol{x^2}$

O	**E**	$\lvert O - E \rvert - 0.5$	$\frac{(\lvert O-E \rvert - 0.5)^2}{E}$
290	300	9.5	$\frac{9.5^2}{300} = 0.301$
310	300	9.5	$\frac{9.5^2}{300} = 0.301$
110	100	9.5	$\frac{9.5^2}{100} = 0.903$
90	100	9.5	$\frac{9.5^2}{100} = 0.903$
		Total	$\boldsymbol{x^2 = 2.408}$

(iii) Level of significance : α=0.05

Degree of Freedom : $(r-1)(c-1) = (2-1)(2-1) = 1$

Critical valve : For 1 degree of freedom at 5% level of significance the table value of $x^2 = 3.81$

Decision : Since the calculated value of $x^2 = 2.408$ is less than the table value of $x^2 = 3.81$, the null hypothesis is accepted.

There is no association between the vaccine and the not affected people.

Q6 (a) Use Gauss divergence theorem to evaluate $\iint \bar{N}.\bar{F}.ds$ where $\bar{F} = x^2 + zj + yzk$ and s is the surface of the cube bounded by x=0, x=1, y=0, y=1, z=0, z=1. (6M)

Solution:

By divergence formula,

$\iint_S \bar{F}.d\bar{S} = \iiint_V \nabla.\bar{F}.div$

Now, $\bar{F} = x^2 + zj + yzk$

$\nabla.\bar{F} = \left(\frac{\delta(x^2)}{\delta x} + \frac{\delta(z)}{\delta y} + \frac{\delta(yz)}{\delta z}\right)$

$= 2x + 0 + y$

$= 2x + y$

$$\iiint_V \nabla.\bar{F}.div = \iiint_V (2x+y).dv = \iiint_V (2x+y).dx.dy.dz$$

$$\iiint_V (2x+y).dx.dy.dz = \int_0^1\int_0^1\int_r^1 (2x+y).dx.dy.dz$$

$= \int_0^1\int_0^1 (2xz + yz|z = 0\ to\ 1)\ .dx\ dy$

$= \int_0^1\int_0^1 (2x + y)\ .dx\ dy$

$= \int_0^1 \left(2xy + \frac{y^2}{2}\middle|y = 0\ to\ 1\right).dx$

$= \int_0^1 (2x + \frac{1}{2}).dx$

$= \left(\frac{2x^2}{2} + \frac{1}{2}x\middle|x = 0\ to\ 1\right)$

$= 1 + \frac{1}{2} = \frac{3}{2}$

$\iint \bar{N}.\bar{F}.ds = \frac{3}{2}$

(b)The mean of two random samples of size 9 and 7 are 196.42 and 198.82 respectively. The sum of the squares of the deviation from the mean are 26.94 and 18.73 respectively. Can the sample be considered to have drawn from the same normal population? (6M)

Solution:

n_1=9 and n_2=7(<30 , so it is small sample)

$\overline{x_1}$=196.42 ; $\overline{x_2}$=198.82 ; $\sum(x_{1i} - \bar{x}_1)^2 = 26.94$; $\sum(x_{2i} - \bar{x}_2)^2 = 18.73$

Step 1 :

Null Hypothesis (H_0): µ1 = µ2 (i.e. Samples are drawn from the same population)

Alternate Hypothesis (H_a): µ1 ≠ µ2 (i.e. Samples are not drawn from the same population)

Step 2:

LOS = 5% (Two tailed test)

Degree of freedom = $n_1 + n_2 - 2 = 9 + 7 - 2 = 14$

Critical value (t_∞) = 2.145

Step 3:

Since Sample is small, sp = $\sqrt{\frac{\sum(x_{1i}-\bar{x}_1)^2+\sum(x_{2i}-\bar{x}_2)^2}{n1 + n2 - 2}} = \sqrt{\frac{26.94+18.73}{9+7-2}} = 1.8061$

S.E. = sp$\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 1.8061\sqrt{\frac{1}{9}+\frac{1}{7}} = 0.9102$

Step 4 : Test Statistic

$t_{cal} = \frac{\overline{x_1}-\overline{x_2}}{S.E.} = \frac{196.42-198.82}{0.9102}$=2.6368

Step 5 :- Decision

Since $|t_{cal}|>t_\infty$, H_0 is rejected ∴∴ The samples cannot be considered to have to have been drawn from the same population.

(c) Find the index, rank, signature and class of the Quadratic Form $x_1{}^2 + 2x_2{}^2 + 3x_3{}^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$ by reducing it to canonical form using congruent transformation method.

Solution: **(8M)**

The matrix form is

$$A=\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 3\end{bmatrix}$$

We write A=IAI

$$\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

By $R_2 - R_1, R_3 - R_1, C_2 - C_1, C_3 - C_1$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 2 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

By $R_3 - 2R_2, C_3 - 2C_2$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 3 & -2 & 1\end{bmatrix} A \begin{bmatrix}1 & -1 & 3\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1 & -1 & 3\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

$x_1 = y_1 - y_2 + y_3$
$x_2 = y_2 - 2y_3$
$x_3 = y_3$

The rank = 3, index = 2

Signature = difference between positive squares and negative squares = 2 – 1 = 1

Since some diagonal elements are positive, some are negative, the value class is indefinite.