M3 SOLUTION OF QUESTION PAPER

(CBCGS MAY 18)

Q.P. Code: 39159

$\boldsymbol{Q.1.a)}$ ***Find Laplace transform of***

$\boldsymbol{f(t) = e^{-t}(\cos 2t . \sin t)}$ **(5)**

Solution: Consider

$$\cos 2t \sin 2t = \frac{1}{2} . 2 \sin t \cos 2t$$

$$= \tfrac{1}{2} [\sin(1+2)t + \sin(1-2)t]$$

$$= \tfrac{1}{2} [\sin 3t - \sin t]$$

$$\therefore L(\cos 2t \sin t) = \frac{1}{2}[L(\sin 3t) - L(\sin t)] = \frac{1}{2} . \left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$$

Now by first shifting theorem,

$$L[e^{-t}(\cos 2t . \sin t)] = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+2s+10} - \frac{1}{s^2+2s+2}\right]$$

$$= \boldsymbol{\frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}}$$

$\boldsymbol{b)}$ ***Show that the set of functions***

$\boldsymbol{\cos nx, n 1, 2, 3 \ldots .}$ ***is orthogonal on*** $\boldsymbol{(0, 2\pi)}$. **(5)**

Solution: *We have* $f_n(x) = \cos nx$.

$$\therefore \int_0^{2\pi} f_m(x).f_n(x)dx = \int_0^{2\pi} \cos mx.\cos nx\, dx$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi}$$

Now, two cases arise.

Case 1: *When* $m \neq n$, *then* $\int_0^{2\pi} f_m(x).f_n(x)dx = 0$

Case 2: *When* $m = n$, *then* $\int_0^{2\pi} f_m(x).f_n(x)dx = \int_0^{2\pi} cos^2\ nx\, dx$

$$\therefore \int_0^{2\pi} [f_n(x)]^2\, dx = \int_0^{2\pi} \left(\frac{1+\cos 2nx}{2}\right) dx = \frac{1}{x}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \pi \neq 0$$

Since, $\int_0^{2\pi} f_m(x).f_n(x)dx \begin{cases} = 0, if\ m \neq n \\ = 0, if\ m = n \end{cases}$

The given set of functions is orthogonal on $[\mathbf{0}, \mathbf{2\pi}]$.

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c) The eqations of lines of regression are $\mathbf{x + 2y = 5}$ ***and*** $\mathbf{2x + 3y = -8}$.

Find 1) means of x and y, 2) coefficient of correlation between x and y. **(5)**

Solution : Lines of regression are $x + 2y = 5$ *and* $2x + 3y = -8$

$$\therefore y = -\frac{1}{2}x + \frac{5}{2} \rightarrow (1) \text{ and } y = -\frac{2}{3}x - \frac{8}{3} \rightarrow (2)$$

$Let\ b_1 = -\frac{1}{2}\ and\ b_2 = -\frac{2}{3}$

$Since\ |b_1| < |b_2|, b_{yx} = b_1 = \frac{-1}{2}\ and\ b_{xy} = \frac{1}{b_2} = -\frac{3}{2} \quad \rightarrow (3)$

$Hence, equation\ (1)\ is\ regression\ equation\ of\ Y\ on\ X - axis\ and$

$equation\ (2)\ is\ regression\ equation\ of\ X\ on\ Y\ type.$

$From\ (1) and\ (2), -\frac{1}{2}x + \frac{5}{2} = -\frac{2}{3}x - \frac{8}{3}$

$\therefore\ \frac{2}{3}x - \frac{1}{2}x = -\frac{8}{3} - \frac{5}{2}$

$\therefore\ \frac{1}{6}x = -\frac{31}{6}$

$\therefore x = -31$

$Substituting\ x = -31\ in\ (2), y = -\frac{2}{3}(-31) - \frac{8}{3}$

$\therefore y = 18$

$Now, r = \pm\sqrt{b_{yx}.b_{xy}}$

$= \pm\sqrt{\frac{-1}{2} \times \frac{-3}{2}}\ \ (from\ 3)$

$= \pm 0.8660$

$Since,, b_{yx}\ and\ b_{xy}\ are\ both\ negative, 'r'\ is\ negative.$

$\therefore r = -0.8660$

$Hence,$

$\boldsymbol{Means\ of\ x\ and\ y: \bar{x} = -31;\ \bar{y} = 7}$

$\boldsymbol{Coefficient\ of\ Correlation\ between\ x\ and\ y\ (r) = -0.8660.}$

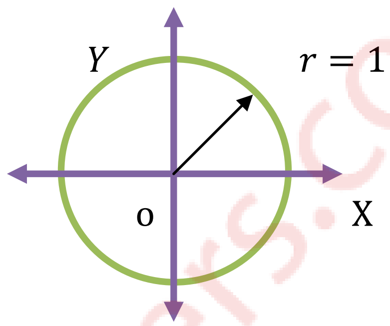
$\boldsymbol{d) \; Evaluate \int_c (z^2 + 2\bar{z} + 1) dz \; where \; C \; is \; the \; circle \; |z|}$ **(5)**

$Solution: \; |z| = 1 \; is \; circle \; with \; centre \; (0, 0) and \; radius = 1$

$Put \; z = re^{i\theta} = 1e^{i\theta}$

$\therefore dz = e^{i\theta}. id\theta$

$and, \bar{z} = e^{-i\theta}$



$Limits \; for \; complete \; circle \; are \; \theta = 0 \; to \; \theta = 2\pi$

$$\therefore \int_c (z^2 - 2\bar{z} + 1) \, dz = \int_0^{2\pi} \left[\left(e^{i\theta}\right)^2 - 2 \, e^{-i\theta} + 1 \right]. ie^{i\theta} d\theta$$

$$= i \int_0^{2\pi} \left[e^{i2\theta} - 2 \, e^{-i\theta} + 1 \right]. ie^{i\theta} d\theta$$

$$= i \int_0^{2\pi} \left[e^{i3\theta} - 2 \, e^{0} + e^{i\theta} \right] d\theta$$

$$= i \left[\frac{e^{i3\theta}}{3i} - 2\theta + \frac{e^{i\theta}}{i} \right]_0^{2\pi}$$

$$= \left[\frac{e^{i3\theta}}{3} - 2i\theta + e^{i\theta} \right]_0^{2\pi}$$

$$= \left[\frac{e^{i6\pi}}{3} - 4i\pi + e^{i2\pi} \right] - \left[\frac{e^{0}}{3} - 0 + e^{0} \right]$$

$$= \left[\frac{1}{3} (\csc 6\pi + i \sin 6\pi) - 4i\pi + (\cos 2\pi + i \sin 2\pi) \right] - \left[\frac{1}{3} + 1 \right]$$

$$= \left[\frac{1}{3} (1 + 0) - 4i\pi + (1 + 0) \right] - \frac{4}{3}$$

$$= \frac{1}{3} - 4i\pi + 1 - \frac{4}{3}$$

$$= -4i\pi$$

$$\boldsymbol{Hence, \int_c (z^2 + 2\bar{z} + 1)dz = -4i\pi}$$

Q.2.a) Using convolution theorem, find the inverse Laplace transform of $\boldsymbol{f(s) = \frac{1}{(s^2+9)(s^2+4)}}$. **(6)**

$$Solution = L^{-1}\left[\frac{\mathbf{1}}{(\boldsymbol{s^2+9})(\boldsymbol{s^2+4})}\right] = L^{-1}\left[\frac{1}{(\boldsymbol{s^2+9})} \times \frac{1}{(\boldsymbol{s^2+4})}\right]$$

$$Let\ \emptyset_1(s) = \frac{1}{s^2+3^2}\ ;\quad \emptyset_2(s) = \frac{1}{s^2+2^2}$$

$$\therefore\ f_1(t) = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}\sin 3t\ and$$

$$f_2(t) = L^{-1}\left[\frac{1}{s^2+2^2}\right] = \frac{1}{2}\sin 2t$$

By Convolution theorem,

$$L^{-1} = [\emptyset_1(s)\emptyset_2(s)] = \int_0^1 f_1(u)f_2(t-u)du$$

$$\therefore\ L^{-1}\left[\frac{1}{(\boldsymbol{s^2+9})} \times \frac{1}{(\boldsymbol{s^2+4})}\right] =$$

$$= \int_0^t \frac{1}{3}\sin 3u.\frac{1}{2}\sin 2(t-u)du \times \frac{2}{2}$$

$$= \frac{1}{12}\int_0^t 2\sin 3u.\sin(2t-2u)du$$

$$= \frac{1}{12}\int_0^t [\cos(3u-2t+2u) - \cos(3u+2t-2u)]\,du$$

$$= \frac{1}{12}\int_0^t [\cos(5u-2t) - \cos(u+2t)]\,du$$

$$= \frac{1}{12}\left[\frac{\sin(5u-2t)}{5} - \sin(u+2t)\right]_0^t$$

$$= \frac{1}{12}\left\{\left[\frac{\sin(5t-2t)}{5} - \sin(t+2t)\right] - \left[\frac{\sin(0-2t)}{5} - \sin(0+2t)\right]\right\}$$

$$= \frac{1}{12}\left\{\left[\frac{\sin 3t}{5} - \sin 3t\right] - \left[\frac{-\sin 2t}{5} - \sin 2t\right]\right\}$$

$$= \frac{1}{12}\left\{\left[\frac{-4\sin 3t}{5}\right] - \left[\frac{-6\sin 2t}{5}\right]\right\}$$

$$= \frac{1}{12}\left\{\frac{-4\sin 3t}{5} + \frac{6\sin 2t}{5}\right\}$$

$$= \frac{1}{12} \times \frac{2}{5}\{-2\sin 3t + 3\sin 2t\}$$

$$= \frac{1}{30}(3\sin 2t - 2\sin 3t)$$

$$\therefore\ \boldsymbol{L^{-1}\left[\frac{1}{(s^2+9)(s^2+4)}\right] = \frac{1}{30}(3\sin 2t - 2\sin 3t)}$$

***b*) *Obtain Fourier series of* $\boldsymbol{f(x) = |x|}$ *in* $\boldsymbol{(-\pi,\pi)}$.** **(6)**

solution : $f(x) = |x|$

$\therefore f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)$ *is even function.*

$\therefore\ b_n = 0$

Here, $l = \pi$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |x| dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi} [\pi^2 - 0]$$

$$= \pi$$

$$a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |x| \cos \frac{n\pi x}{\pi} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x . \frac{\sin nx}{n} - 1 . \frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\pi . \frac{\sin n\pi}{n} + \frac{\cos nx}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi} \left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

In Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\therefore \; |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \left[[(-1)^n - 1]\right] \cos \frac{n\pi x}{\pi} + 0$$

$$\therefore\ |x| = \frac{\pi}{2} + \frac{2}{\pi}\left(\frac{-2\cos x}{1^2} + 0 - \frac{2\cos 3x}{3^2} + 0 - \frac{2\cos 5x}{5^2} + \cdots..\right)$$

$$\therefore\ |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots..\right) \rightarrow (1)$$

Deduction : Put $x = 0$ *in* (1)

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \cdots..\right)$$

$$\therefore\ -\frac{\pi}{2} = -\frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\therefore\ \frac{-\pi}{2} \times \frac{\pi}{-4} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\therefore\ \boldsymbol{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

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c) *Find the bilinear transformation which maps the points* $\boldsymbol{z = 1}$**,** $\boldsymbol{i, -1}$**. *Onto the points*** $\boldsymbol{w = i, 0, -i}$**. *Hence, find the image of*** $|z| < 1$ ***onto the*** $\boldsymbol{w}$ ***− plane.*** **(8)**

Solution : Part : 1

Let the bilinear transformation be $w = \dfrac{az + b}{cz + d} \rightarrow (1)$

(*where* a, b, c, d *are complex constants and* $ad - bc \neq 0$)

Put $z = 1$ *and* $w = i$ *in* (1)

$$\therefore i = \frac{a(1) + b}{c(1) + d}$$

$$\therefore ic + id = a + b \rightarrow (2)$$

Put $z = i$ *and* $w = 0$ *in* (1)

$$\therefore 0 = \frac{a(i) + b}{c(i) + d}$$

$\therefore 0 = ai + b$

$\therefore b = -ai \quad \rightarrow (3)$

$Put\ z = -1\ and\ w = -1\ in\ (1)$

$$\therefore\ -i = \frac{a(-i) + b}{c(-i) + d}$$

$\therefore ic - id = -a + b \quad \rightarrow (4)$

$Adding\ (2) and\ (4), we\ get, 2\ ic = 2b$

$$\therefore c = \frac{b}{i} = \frac{-ai}{i} = -a \quad \rightarrow (5)(From\ 1)$$

$Substituting\ (3) and\ (5) in\ (2) we\ get,$

$\therefore\ -ia + id = a - ia$

$\therefore id = a$

$$\therefore d = \frac{a}{i} = -ai \quad \rightarrow (6)$$

$Substituting\ (3), (5)\ and\ (6) in\ (1), we\ get,$

$$\therefore w = \frac{az - ai}{-az - ai} = \frac{-a(-z + i)}{-a\,(z + i)}$$

$$\therefore w = \frac{(-z + i)}{(z + i)}$$

$$\therefore w = \frac{i - z}{z + i} \quad \rightarrow (7) is\ the\ Bilinear\ Transformation.$$

$Part\ 2:$

$For\ fixed\ points, put\ w = z\ in\ (7)$

$$\therefore z = \frac{i - z}{z + i}$$

$\therefore\ z^2 + iz = i - z$

$\therefore\ z^2 + z(i + 1) + z - i = 0$

$$\therefore z = \frac{-(i+1) \pm \sqrt{(i+1)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore z = \frac{-(i+1) \pm \sqrt{6i}}{2}, are\ the\ fixed\ points.$$

Since the fixed points are not equal, the bilinear transformation is not Parabolic.

Part : 3

From (7), $\quad wz + wi = i + z$

$\therefore\ wz + w = i - iw$

$\therefore z(w+1) = i - iw$

$$\therefore z = \frac{i(1-w)}{w+1}$$

Given, $|z| \leq 1$, *which is the interior of a circle with centre* $(0,0)$ *and radius* $= 1$ *unit in* $z-$ *plane.*

$$\therefore\ |z| \leq 1\ \Rightarrow \left|\frac{i(1-w)}{w+1}\right| \leq 1$$

$\therefore\ |i|.|1-w| \leq |w+1|$

$\therefore\ |0+1i|.|1-(u+iv)| \leq |u+iv+1|$

$\therefore\ \sqrt{0+1^2}.\sqrt{(1-u)^2+(-v)^2}\ \leq\ \sqrt{(1+u)^2+v^2}$

On squaring, $1.(1-2u+u^2+v^2) \leq (1+2u+u^2+v^2)$

$\therefore 1 - 2u + u^2 + v^2\ \leq 1 + 2u + u^2 + v^2$

$\therefore 0\ \leq 4u$

$\therefore u \geq 0$, *which is right half of* $v-$ *axix in* $w-$ *plane.*

Hence, the interior of the circle $|z| \leq 1$ *in* $z-$ *plane is mapped onto the right half of* $v-$ *axis in* $w-$ *plane.*

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3. a) If $v = e^x \sin y$, prove that v is a harmonic function. Also find the corresponding harmonic conjugate function and analytic function. **(6)**

Solution: We have $\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial^2 v}{\partial x^2} = e^x \sin y$

$\frac{\partial v}{\partial y} = e^x \cos y \qquad \frac{\partial^2 v}{\partial y^2} = -e^x \sin y$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \therefore$ *v satisfies Laplace's equation.*

Now, we use, Milne − Thompson Method .

$v_x = e^x \sin y \quad \therefore \varphi 2(z,0) = 0; \quad v_y = e^x \cos y \quad \therefore \varphi 1\,(z,0) = e^z$

$\therefore f'(z) = \varphi 1\,(z,0) + i\varphi 2(z,0) = e^z + 0$

$\therefore f(z) = e^z + c$

Now, $f(z) = e^z = e^{x+iy} = e^x + e^{iy} = e^x(\cos y + i \sin y)$

$\therefore \boldsymbol{u = e^x \cos y}$

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b) Using Bender − Schmidt method, solve $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$, subject to the conditions, $u(0,t) = 0, u(5,t) = 0, u(x,0) = x^2(25 - x^2)$ taking $h = 1$, for 3 minutes. **(6)**

Solution : We have given $h = 1$ and $a = 1$

$k = \frac{a}{2} h^2 \qquad \therefore k = \frac{1}{2}.(1)^2 = 0.5$

Since, $h = 1$, *and the x is* 0 *to* 5. *We divide x interval into* 5 *parts by taking* $h = 1$.

We also divide the time interval by taking $k = 0.5$ *upto*3.

$t_0 = 0, t_1 = 0.5, t_2 = 1.0, t_3 = 1.5, t_4 = 2, t_5 = 2.5, t_6 = 3.$

$By\ data, \qquad u(0,t) = 0$

$Hence, for\ all\ x = 0\ and\ t = 0, 0.5, 1, 1.5, 2, 2.5, 3.$

$\therefore\ u(0,t) = 0\ for\ all\ t.$

$By\ data \qquad u(5,t) = 0$

$Hence\ for\ all\ x = 5\ and\ t = 0, 0.5, 1, 1.5, 2, 2.5, 3.$

$\therefore u(5,t) = 0\ for\ all\ t.$

$Now, u(x,0) = x^2(25 - x^2)$

$We\ now\ calculate\ u\ for\ t = 0\ and\ x = 1, 2, 3, 4, 5.$

$When\ x = 0, t = 0, \qquad u = 0^2(25 - 0^2) = 0$

$x = 1, t = 0, \qquad u = 1^2(25 - 1^2) = 24$

$x = 2, t = 0, \qquad u = 2^2(25 - 2^2) = 84$

$x = 3, t = 0, \qquad u = 3^2(25 - 3^2) = 144$

$x = 4, t = 0, \qquad u = 4^2(25 - 4^2) = 144$

$x = 5, t = 0, \qquad u = 5^2(25 - 5^2) = 0$

t \ x	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.06	32.25	21.75	0

$\boldsymbol{Used\ Bender\ Schmidt\ formula\ c = \frac{a+b}{2}\ for\ remaining\ values.}$

$c)$ ***Using Residue theorem, evaluate*** **(8)**

1) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ 2) $\int_0^{\infty} \frac{dx}{x^2+1}$

$\boldsymbol{Q.4.a)}$ ***Solve by Crank − Nicholson simplified formula***

$$\boldsymbol{\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, u(0,t) = 0, u(1,t) = 2t, u(x,0) = 0\ taking\ h = 0.25}$$

for two − time steps. **(6)**

Solution: *Here we have* $a = 1$.

Since the interval of x *is* 0 *to* 1, *choose* $h = 0.25$.

$\therefore\ x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1.$

Since to use simplified formula of Crank − Nicholson, we must have

$k = ah^2$, *we take* $k = 1.\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Since we want the values of u *for two steps only we take* $k = \frac{1}{16}, \frac{2}{16}$.

$\therefore\ t_0 = 0, t_1 = \frac{1}{16}, t_2 = \frac{2}{16}$

By data $u(x,0) = 0$.*e*., *for all values of* x *when* $t = 0, u = 0$;

when $x = 0.00, 0.25, 0.50, 0.75, 1.00$

By data $u(0,t) = 0$ *i.e.*, *when* $x = 0$, *for all values of* $t, u = 0$,

when $t = 0, \frac{1}{16}, \frac{2}{16}$.

$By\ data\ u(1,t) = 2t\ i.e., when\ x = 1, and\ t = 0, u = 0;$

$$When\ x = 1\ and\ t = \frac{1}{16}, \quad u = 2\left(\frac{1}{16}\right) = \frac{1}{8}$$

$$when\ x = 1\ and\ t = \frac{2}{16}, \quad u = 2\left(\frac{2}{16}\right) = \frac{1}{4}.$$

t \ x	0.00	0.25	0.25	0.75	1.00
0	0	0	0	0	0
1/16	0	u_1	u_2	u_3	1/8
2/16	0	u_4	u_5	u_6	1/4

$Now, by\ Crank - Nicholson\ formula\ we\ calculate\ the\ remaining\ values.$

$$e = \frac{1}{4}(a + b + c + d) \quad \rightarrow (1)$$

$$u_1 = \frac{1}{4}(0 + 0 + 0 + u_2) = \frac{u_2}{4} \quad \rightarrow (2)$$

$$u_2 = \frac{1}{4}(0 + 0 + u_1 + u_3) = \frac{1}{4}(u_1 + u_3) \quad \rightarrow (3)$$

$$u_3 = \frac{1}{4}\left(0 + 0 + u_2 + \frac{1}{8}\right) = \frac{1}{4}\left(u_2 + \frac{1}{8}\right) \quad \rightarrow (4)$$

$We\ solve\ these\ equations\ to\ obtain\ u_1, u_2, u_3.$

t \ x	0.00	0.25	0.50	0.75	1.00
0	0	0	0	0	0
1/16	0	1/448	1/112	15/448	1/8
2/16	0	u_4	u_5	u_6	1/4

$We\ repeat\ the\ above\ process\ again. By\ equation\ (1).$

$$u_4 = \frac{1}{4}\left(0 + \frac{1}{112} + 0 + u_5\right)$$

$$= \frac{1}{4}\left(\frac{1}{112} + u_5\right) = \frac{1}{4}(0.0089 + u_5) \quad \rightarrow (5)$$

$$u_5 = \frac{1}{4}\left(\frac{1}{448} + \frac{15}{448} + u_4 + u_6\right)$$

$$= \frac{1}{4}(0.0022 + 0.0335 + u_4 + u_6) = \frac{1}{4}(0.0357 + u_4 + u_6) \quad \rightarrow (6)$$

$$u_6 = \frac{1}{4}\left(\frac{1}{112} + \frac{1}{8} + u_5 + \frac{1}{4}\right)$$

$$= \frac{1}{4}(0.0089 + 0.1250 + u_5 + 0.250)$$

$$= \frac{1}{4}(0.3839 + u_5) \quad \rightarrow (7)$$

Putting the values of u_4 *and* u_6 *from* (5) *and* (7)*in* (6) *we get*.

t \ x	0.00	0.25	0.50	0.75	1.00
0	0.00	0.00	0.00	0.00	0.00
1/16	0.00	0.0022	0.0089	0.0334	0.125
2/16	0.00	0.0117	0.03822	0.1080	0.250

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b) Obtain the Taylor's and Laurent series which represent the

the function $f(z) = \dfrac{2}{(z-1)(z-2)}$ ***in the regions,***

1) $|z| = 1$ **2)** $1 < |z| < 2$. (6)

Solution: Let $\dfrac{2}{(z-1)(z-2)} = \dfrac{a}{z-1} + \dfrac{b}{z-2}$

$2 = a(z-2) + b(z-1)$

When $z = 1$, $2 = -a$ $\therefore$ $a = -2$

When $z = 2$, $2 = b$

$$\therefore \quad \frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}$$

Case(1): *When* $|z| < 1$, *clearly* $|z| < 2$

$$\therefore f(z) = \frac{2}{1-z} - \frac{2}{2[1-(1-z/2)]} = 2[1-z]^{-1} - [1-z/2]^{-1}$$

$$\therefore \boldsymbol{f(z) = 2[1 + z + z^2 + z^3 + \cdots] - \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]}$$

Case(2): *When* $1 < |z| < 2$, *we write*

$$\frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2} \ as$$

$$= -\frac{2}{z[1-(1-z)]} - \frac{2}{2[1-(1-z/2)]}$$

$$= -\frac{2}{z}[1-(1/z)]^{-1} - [1-(z/2)]^{-1}$$

$$\boldsymbol{f(z) = -\frac{2}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right] - \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]}$$

---

c) *Solve* $\boldsymbol{(D^2 - 3D + 2)y = 4e^{2t}}$ ***with*** $\boldsymbol{y(0) = -3, y'(0) = 5}$ ***where***

$\boldsymbol{D = \frac{d}{dt}}$. **(8)**

Solution: *Let* $L(y) = \bar{y}$. *Then, taking Laplace transform,*

$L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$

But $L(y') = s\bar{y} - y(0) = s\bar{y} + 3$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} + 3s - 5$

∴ *The equation becomes,*

$$(s^2\bar{y} + 3s - 5) - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} = \frac{4}{s-2} + 14 - 3s = \frac{-3s^2 + 20s - 24}{s-2}$$

$$\therefore \ \bar{y} = \frac{-3s^2 + 20s - 24}{(s-2)(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

$By\ partial\ fraction,\quad \bar{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$Taking\ inverse\ Laplace\ transform,$

$$y = -7L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\left(\frac{1}{s-2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$= -7e^{t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s^2}$$

$\boldsymbol{Hence, y = -7e^{t} + 4e^{2t} + 4te^{2t}}.$

$\boldsymbol{Q.5.a)\ Find\ an\ analytic\ function\ f(z) = u + iv, if}$

$\boldsymbol{u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\sin y\}}$ **(6)**

$Solution:\ Let\ u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\ \sin y\}$

$$\therefore\ u_x = -e^{-x}\{(x^2 - y^2)\cos y + 2xy\ \sin y\} + e^{-x}\{2x\cos y + 2y\ \sin y\}$$

$$= e^{-x}[-(x^2 - y^2)\cos y + 2x\cos y + 2y\sin y - 2xysin\ y]$$

$$u_y = e^{-x}[-(x^2 - y^2)\sin y - 2y\cos y + 2x\sin y - 2xy\ cos\ y]$$

$\therefore\ \varphi_1 = u_x,\quad and\quad \varphi_2 = u_y$

$By\ Milne - Thompson\ method$

$$f'(z) = \varphi_1(z,0) - i\varphi_2\,(z,0) = e^{-z}[-z^2 + 2z]$$

$$\therefore f(z) = \int e^{-z}[-z^2 + 2z]dz$$

$Integrating\ by\ parts,$

$$f(z) = (-z^2 + 2z)(e^{-z}) - \int(-e^{-z})(-2z + 2)dz$$

$$= e^{-z}(z^2 - 2z) + \int e^{-z}(2 - 2z)dz$$

$Integrating\ by\ parts\ again,$

$$\therefore f(z) = e^{-z}(c - 2z) + (2 - 2z)(-e^{-z}) - \int (-e^{-z})(-2)dz$$

$$= e^{-z}(z^2 - 2z) - e^{-z}(2 - 2z) + 2e^{-z}$$

$$= \boldsymbol{e^{-z}e^{-z} + c}.$$

---

b) Find the Laplace transform of $\frac{\sin at}{t}$***. Does the Laplace transform of*** $\frac{\cos at}{t}$ ***exist?*** **(6)**

Solution : Consider $f(t) = \sin at$

$$\therefore \quad L[f(t)] = \frac{a}{s^2 + a^2} = \phi(s)$$

$$\therefore \quad L\left[\frac{\sin at}{t}\right] = \int_s^\infty \phi(s)ds$$

$$= \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}.$$

Now consider, $f(t) = \cos t\, at$

$$\therefore \quad L[f(t)] = \frac{s}{s^2 + a^2} = \phi(s)$$

$$\therefore \quad L\left[\frac{\cos at}{t}\right] = \int_s^\infty \phi(s)ds = \int_s^\infty \frac{s}{s^2 + a^2} ds = \frac{1}{2}[\log(s^2 + a^2)]_s^\infty$$

Since $\boldsymbol{\log(s^2 + a^2)}$ ***is infinite when*** $\boldsymbol{s \to \infty, L\left[\frac{\cos at}{t}\right]}$ ***does not exist.***

---

c) Obtain half range Fourier cosine series of $\boldsymbol{f(x) = x, 0 < x < 2}$.

Using Parseval's identity , deduce that –

$$\boldsymbol{\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots} \qquad \textbf{(8)}$$

$Solution: \; Let \; f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right). \qquad Here, l = 2.$

$$\therefore \; a_0 = \frac{1}{l}\int_0^l f(x)dx = \frac{1}{2}\int_0^2 xdx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \left[x\frac{\sin(n\pi x/2)}{n\pi/2} + \frac{\cos(n\pi x/2)}{n^2\pi^2/2^2}.1\right]_0^2$$

$$\therefore \; a_0 = \left[2.(0) + \frac{\cos n\pi}{n^2\pi^2/2^2} - 0 - \frac{1}{n^2\pi^2/2^2}\right]$$

$$= \frac{[(-1)^n - 1]}{n^2\pi^2/2^2}$$

$$= \left\{-4.\frac{2}{n^2\pi^2} \quad if \; n \; is \; odd\right.$$

$$= \{0 \qquad if \; n \; is \; even$$

$$\therefore \; x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right]$$

$By \; Parseval's \; identity$

$$\frac{1}{l}\int_0^l [f(x)]^2dx = \frac{1}{2}[2a_0^2 + a_1^2 + a_2^2 + \cdots]$$

$$\therefore l.h.s. = \frac{1}{2}\int_0^2 x^2dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

$$\therefore \; \frac{4}{3} = \frac{1}{2}\left[2 + \frac{64}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right\}\right]$$

$$\frac{8}{2} - 2 = \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$\therefore \ \dfrac{\pi^4}{96} = \dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \cdots$

$Q.6.a)$ Obtain Complex form of Fourier series of

$\boldsymbol{f(x) = e^x}$, $-\mathbf{1} < x < 1.$ (6)

Solution : Let $c = -1$ and $c + 2l = 1$

$\therefore -1 + 2l = 1$

$\therefore 2l = 2$

$\therefore l = 1$

Given, $f(x) = e^x$

$$C_n = \frac{1}{2l}\int_c^{c+2l} f(x)e^{-inx/l}\,dx$$

$$= \frac{1}{2(1)}\int_{-1}^{1} e^x \ e^{-inx/1}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{x-in\pi x}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{(1-in\pi)x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{(1-in\pi)x}}{1-inx}\right]_{-1}^{1}$$

$$= \frac{1}{2} \times \frac{1}{1-inx}\left[e^{(1-in\pi)} - e^{-(1-in\pi)}\right]$$

$$= \frac{1}{2(1-inx)} \times \left[e^1 e^{-in\pi} - e^{-1}e^{1-in\pi}\right] \times \frac{(1+in\pi)}{(1+in\pi)}$$

Consider,

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n \pm i0 = (-1)^n$$

$$\therefore C_n = \frac{(1 + in\pi)}{2(1^2 - i^2n^2\pi^2)} \times [e^1(-1)^n - e^{-1}(-1)^n]$$

$$= \frac{(1 + in\pi)}{2(1 + n^2\pi^2)}(-1)^n \times [e^1 - e^{-1}]$$

$$= \frac{(1 + in\pi)}{2(1 + n^2\pi^2)}(-1)^n \times 2 \sin h1$$

In Coplex Fourier Series,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/_1}$$

$$\therefore\ e^x = \sum_{n=-\infty}^{\infty} \frac{(1 + in\pi x)}{(1 + n^2\pi^2)} \sin h1 . e^{in\pi x/_1}$$

$$\therefore\ \boldsymbol{e^x = \sin h1 \sum_{n=-\infty}^{\infty} \frac{(1 + in\pi x)}{(1 + n^2\pi^2)} (-1)^n . e^{in\pi x/_1}}$$

---

b) Fit a straight line to the following data, **(6)**

X	100	120	140	160	180	200
y	0.45	0.55	0.60	0.70	0.80	0.85

Solution :

x	y	$X = x - 150$	$Y = y$	X^2	XY
100	0.45	-50	0.45	2500	-22.5
120	0.55	-30	0.55	900	-16.5
140	0.60	-10	0.60	100	-6.0
160	0.70	10	0.70	100	7.0
180	0.80	30	0.80	900	24.0
200	0.85	50	0.85	2500	42.5
	TOTAL	0	3.95	7000	28.5

$Here, n = 6$

$Let\ the\ equation\ of\ straight\ line\ by\ Y = a + bX \quad \rightarrow (1)$

$Subject\ to,$

$$\sum Y = an + b \sum X$$

$\therefore 3.95 = 6a + 0$

$\therefore a = 0.6853$

$$And, \quad \sum XY = a \sum X + b \sum X^2$$

$\therefore 28.5 = 0 + 7000b$

$\therefore b = 0.004071$

$\therefore From\ (1), the\ equation\ of\ straight\ line\ is\ Y = 0.6583 + 0.00407X$

$Re - substitute\ 'X'\ and\ 'Y', y = 0.6583 + 0.004071(x - 150)$

$\therefore y = 0.6583 + 0.004071x - 0.6107$

$\therefore y = 0.0476 + 0.004071x$

$\therefore \boldsymbol{The\ equation\ of\ straight\ line\ fit\ is\ y = 0.0476 + 0.004071x.}$

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c) A string is stretched and fastened to two points distance l apart. Motion is started by displacing the string in form $y = a\sin(n\pi/l)$ from which it is released at a time $t = 0$. If the vibrations of a string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, show that the displacement of a point at a distance x from one end at time t is given by

$y(x,t) = a\sin\left(\frac{n\pi}{l}\right)\cos\left(\frac{\pi ct}{l}\right)$. **(8)**

$Solution : The\ vibration\ of\ a\ string\ is\ given\ by\ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \rightarrow (1)$

$Since\ the\ vibration\ of\ a\ string\ is\ periodic\ the\ solution\ of\ (1) is\ of\ the\ form$

$$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4\ \sin mct) \quad \rightarrow (2)$$

$Given\ initial\ boundary\ conditions\ are :$

1) $When\ x = 0, y = 0\ for\ all\ t\ i.e. one\ end\ A\ of\ the\ string\ remains\ fixed$

$throughout\ the\ motion : We\ get, from\ (2)$

$$0 = (c_1 + 0)(c_3 \cos mct + c_4 \sin mct) \qquad \therefore c_1 = 0$$

$$\therefore y = c_2 \sin mx(c_3\ \cos mct + c_4\ \sin mct) \quad \rightarrow (3)$$

2) $Now, \frac{\partial y}{\partial t} = 0\ when\ t = 0\ i.e. when\ initially\ the\ string\ is\ steady\ .$

$$from\ (3\ ) \frac{\partial y}{\partial t} = c_2 \sin mx\{c_3(-mc) \sin mct + c_4(mc) \cos mct\}$$

$$Putting\ t = 0, \frac{\partial y}{\partial t} = 0, \;\; 0 = c_2 \sin mx(c_4 mc)$$

$$\therefore\ c_2 c_4 mc = 0.$$

$If\ c_2 = 0\ then\ (3) will\ give\ a\ trivial\ solution\ y = 0.$

$$\therefore\ c_4 = 0.$$

$Thus, from\ (3), we\ get$

$$y = c_2 c_3 \sin mx \cos mct$$

$$= c_5 \sin mx \cos mct \qquad (where\ c_2 c_3 = c_5) \quad \rightarrow (4)$$

3) $Now, y = 0\ when\ x = l\ for\ all\ t\ i.e. the\ other\ end\ of\ the\ string\ is$

$fixed\ and\ the\ length\ of\ the\ string\ is\ l,$

$\therefore From\ (4), we\ get,$

$$\therefore 0 = c_5 \sin ml \cos mct \quad \rightarrow (5)$$

$If\ c_5\ is\ zero, (4) will\ lead\ us\ to\ a\ trivial\ solution\ y = 0, Thus\ , c_5\ cannot$

$be\ zero. Hence, from\ (5) we\ get\ , \sin ml = 0.$

$\therefore ml = n\pi\ \ i.e.\ \ m = \frac{n\pi}{l};\quad n = 1,2,3, ..$

$Hence, from\ (4), we\ get, y = c_5 \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}.$

$Putting\ n = 1,2,3 \ldots. the\ general\ solution\ is$

$$y = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \quad \rightarrow (6)$$

4) $When\ t = 0, y = a \sin\frac{\pi x}{l}\ i.e. initially\ the\ string\ is\ given\ the\ shape\ of$

$the\ curve\ y = a \sin\frac{\pi x}{l}.$

$\therefore When\ t = 0, from (6), we\ get$

$$a \sin\frac{\pi x}{l} = \sum b_n \sin\frac{n\pi x}{l}$$

$\therefore when\ n = 1, b_1 = a\ and\ when\ n = 2,3, \ldots., b_2 = b_3 = \cdots = 0$

$Hence, the\ required\ solution\ is\ from\ (6)$

$$y_{(x,t)} = a \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l}.$$
