MUMBAI UNIVERSITY PAPER SOLUTIONS

SEM II APPLIED MATHS II CBCGS DEC 2019

Q.P. Code: 29701

Q1)a) Evaluate $\int_0^\infty xe^{-x^4}dx$ **.** **(3M)**

Ans : Putting $x^4 = t$

$$x = t^{\frac{1}{4}}$$

$$dx = \frac{1}{4}t^{\frac{-3}{4}}dt$$

$$I = \int_0^\infty t^{\frac{1}{4}}.e^{-t}.\frac{1}{4}t^{\frac{-3}{4}}dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t}.t^{\frac{-1}{2}}dt = \frac{1}{4}\sqrt{\pi}$$

Q1)b) Find the length of the arc of the curve $r = a\sin^2\left(\frac{\theta}{2}\right)$ **from Θ = 0 to any point P(Θ).** **(3M)**

Ans : The required arc is given by

$$L = \int_0^{P(\theta)} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$L = \int_0^{P(\theta)} \sqrt{r^2 + \left(X\frac{1}{2}a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta$$

$$L = \int_0^{P(\theta)} \sqrt{r^2 + \left(X\frac{1}{2}a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)}d\theta$$

$$= \int_0^{P(\theta)} \sqrt{r^2 + ar\ \ os^2\frac{\theta}{2}}d\theta = \int_0^{P(\theta)} r\sqrt{1+\frac{a}{r}\cos^2\frac{\theta}{2}}d\theta$$

$$= \frac{r}{2}\left[\frac{1}{\sqrt{1+\frac{a}{r}\cos^2\frac{\theta}{2}}}2ar\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right]_0^{P(\theta)}$$

Q1)c) Solve $(D^4 - 2D^2 + 1)y = 0$. **(3M)**

Ans : The auxiliary equation is

$$D^4 - 2D^2 + 1 = 0$$
$$\therefore \left(D^2 - 1\right)^2 = 0$$
$$\therefore D^2 - 1 = 0$$
$$\therefore D^2 = 1,1$$
$$\therefore D = 1,-1,1,-1$$

The roots are real and repeated .

Therefore, $y = \left(c_1 + c_2 x\right)e^x + \left(c_3 + c_4 x\right)e^{-x}$

Q1)d) Solve $(x - 2e^y)dy + (y + x\sin x)dx = 0$. **(3M)**

Ans : We have M = y + xsinx and N = x – 2e^y

$$\therefore \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$
$$\therefore \int M dx = \int \left(y + x\sin x\right)dx = yx + x(-\cos x) - \int(-\cos x).1.dx$$
$$= xy - x\cos x + \sin x$$
$$\therefore \int -2e^y dy = -2e^y$$

The solution $xy - x\cos x + \sin x - 2e^y = c$.

Q1)e) Evaluate $\int_0^1 \int_0^x x^2 y^2 (x + y) dy dx$. **(4M)**

Ans :

$$\int_0^1 \int_0^x (x^3 y^2 + x^2 y^3) dy dx$$
$$= \int_0^1 \left[\frac{x^3 y^3}{3} + \frac{x^2 y^4}{4}\right]_0^x dx$$
$$= \int_0^1 \left(\frac{x^6}{3} + \frac{x^6}{4}\right)dx = \frac{7}{12}\int_0^1 x^6 dx = \left[\frac{7}{12} X \frac{x^7}{7}\right]_0^1 = \frac{1}{12}$$

Q1)f) Solve $\frac{dy}{dx} = x^3 + y$ with initial conditions $x_0 = 1$, $y_0 = 1$ by Taylor's method. Find the approximate value of y for x=0.1. **(4M)**

Ans : The Taylor's Series is given by

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + ...$$

Now,

$$y' = x^3 + y$$
$$y_0' = x_0^3 + y_0 = 1^3 + 1 = 1$$
$$y'' = 3x^2 + y'$$
$$y_0'' = 3x_0^2 + y_0' = 3 + 1 = 4$$
$$y''' = 6x + y''$$
$$y_0''' = 6x_0 + y_0'' = 6 + 4 = 10$$

Putting these values in the series , we get

y= 1 + 0.1 X 1 + (0.01/2) X 4 + (0.001/6) X 10 + ...

y =1.12167

The approximate value of y is 1.12167 .

Q2)a) Solve $\frac{d^2y}{dx^2} - 4y = x^2e^{3x} + e^{3x} - \sin 2x$. **(6M)**

Ans : The auxiliary equation is D^2-4=0 . Hence, D = 2, -2 .

The Complementary Equation is $y = c_1e^{2x} + c_2e^{-2x}$.

$$P.I. = \frac{1}{D^2 - 4}\left(x^2e^{3x} + e^{3x} - \sin 2x\right) .$$

$$\frac{1}{D^2-4}e^{3x}x^2 = e^{3x}.\frac{1}{D^2-4}.x^2 = e^{3x}.\frac{1}{(D+3)^2-4}.x^2 = e^{3x}.\frac{1}{D^2+6D+5}.x^2 .$$

$$= \frac{e^{3x}}{5}\left[1 + \frac{D^2+6D}{5}\right]^{-1}x^2 = \frac{e^{3x}}{5}\left[1 - \frac{D^2+6D}{5} + \frac{36D^2}{25} + ...\right]x^2$$

$$= \frac{e^{3x}}{5}\left[x^2 - \frac{12x}{5} - \frac{2}{5} + \frac{72}{25}\right] = \frac{e^{3x}}{5}\left[x^2 - \frac{12x}{5} + \frac{62}{25}\right]$$

$$\frac{1}{D^2-4}e^x = \frac{-1}{3}e^x$$

$$\frac{1}{D^2-4}\sin 2x = -\frac{1}{2}\sin 2x$$

The complete solution is

$$y = c_1e^{2x} + c_2e^{-2x} + \frac{e^{3x}}{5}\left[x^2 - \frac{12x}{5} + \frac{62}{25}\right] + \frac{-1}{3}e^x - \frac{1}{2}\sin 2x\,.$$

Q2)b) Show that $\int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a}$ **, (a>0)** **(6M)**

Ans : Let I(a) be the given integral. Then by the rule of differentiation under integration sign,

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}dx = \int_0^\infty \frac{1}{x^2}.\frac{1}{1+ax^2}.x^2dx = \int_0^\infty \frac{dx}{1+ax^2}$$

$$= \frac{1}{a}\int_0^\infty \frac{dx}{(1/a)+x^2} = \frac{1}{a}.(\sqrt{a})\left[\tan^{-1}x\sqrt{a}\right]_0^\infty = \frac{1}{\sqrt{a}}.\frac{\pi}{2}$$

$$\therefore \frac{dI}{da} = \frac{\pi}{2\sqrt{a}}$$

Integrating both sides , $I(a) = \frac{\pi}{2}\int \frac{da}{\sqrt{a}} = \pi\sqrt{a} + c$.

To find c , put a= 0. Hence, I(0)=c .

But

$$I(0) = \int_0^\infty 0dx = 0, \therefore c = 0, \therefore I = \pi\sqrt{a}$$

$$\therefore \int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a}.$$

Q2)c) Change the order of integration and evaluate $\int_0^5 \int_{2-x}^{x+2} dydx$ **.** **(8M)**

Ans : 1.Given order and given limits : Given order is first w.r.t y and then w.r.t x ,i.e. a strip parallel to the y-axis. Y changes from y=2-x to y=2+x and then x changes from x=0 to x=5.

2. Region of integration : y=2-x is a straight line x+y=2 and y=2+x is also a straight line. x=0 is the y-axis and x=5 is a line parallel to the y-axis .The points of intersection are A(0,2), B(5,-3) and C(5,7)

.The region of integration is the triangle ABC .

3. Change the order of integration : To change the order of integration, consider a strip parallel to the x-axis in the region of integration .When the strip moves parallel to itself, its base moves on two different straight lines AB and AC .Thus, the region of integration is split into two parts, ADC and ADB .So we consider two strips in the two regions. In the region ABD on the strip x varies from x= 2-x to x=5 and then the strip moves from y =-3 to y=2 .In the region ADC, on the strip x varies from x= y-2 to x =5 and then the strip moves from y =2 to y =7.

$$\therefore I = \int_{-3}^{2}\int_{2-y}^{5} dxdy + \int_{2}^{7}\int_{y-2}^{5} dxdy$$

$$= \int_{-3}^{2}[x]_{2-y}^{5} dy + \int_{2}^{7}[x]_{y-2}^{5} dy$$

$$= \int_{-3}^{2}(3+y)dy + \int_{2}^{7}(7-y)dy \quad .$$

$$= \left[3y + \frac{y^2}{2}\right]_{-3}^{2} + \left[7y - \frac{y^2}{2}\right]_{7}^{2}$$

$$= \left(17 - \frac{9}{2}\right) + \left(37 - \frac{49}{2}\right) = 25$$

Q3)a) Evaluate $\iiint zdxdydz$ **over the volume of tetrahedron bounded by the planes x=0,y=0,z=0 and** $\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=1$ **.** **(6M)**

Ans : We put x=3u , y=4v , z=5w , dx=3du , dy=4dv , dz=5dw .

$$\therefore I = 60\iiint 5wdudvdw \ .$$

As before the limits of integration change and we have

$$\therefore I = 3X\,4X\,5^2 \int_{u=0}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} wdwdvdu$$

$$= 300\int_{u=0}^{1}\int_{v=0}^{1-u}\left[\frac{w^2}{2}\right]_0^{1-u-v} dvdu = \frac{300}{2}\int_{u=0}^{1}\int_{v=0}^{1}(1-u-v)^2\,dvdx$$

$$= 150\int_{u=0}^{1}\left[-\frac{(1-u-v)^3}{3}\right]_0^{1-u} du = -\frac{150}{3}\int_0^1\left[0-(1-u)^3\right]du$$

$$= 50\int_0^1(1-u)^3du = 50\left[-\frac{(1-u)^4}{4}\right]_0^1 = -\frac{50}{4}[0-1] = \frac{25}{2}$$

Q3)b) Find the mass of the lamina bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ if the density of the lamina at any point varies as the square of its distance from the origin. **(6M)**

Ans : The two curves intersect at A(a,a). The lamina is the area OBACO. On the curve OCA, $y = \sqrt{4x} = 2\sqrt{x}$ and on the curve OBA , $y = \frac{x^2}{4}$. The surface density is given by $\rho = k(x^2 + y^2)$ .Taking the elementary strip parallel to the y-axis , on the strip y varies from $y = \frac{x^2}{4}$ to $y = \sqrt{4x} = 2\sqrt{x}$ and then x varies from x=0 to x=4 .

Therefore Mass of the lamina =

$$= k\int_0^4 \int_{x^2/4}^{\sqrt{4x}} (x^2 + y^2) dxdy = k\int_0^a \left[x^2 y + \frac{y^2}{3} \right]_{x^2/4}^{\sqrt{4x}} dx$$

$$= k\int_0^a \left(x^2.\sqrt{4x} + (4x).\sqrt{4x} - x^2.\frac{x^2}{4} - \frac{1}{3}.\frac{x^6}{4^3} \right) dx$$

$$= k\int_0^a \left(2x^{5/2} + \frac{4.2}{3}.x^{3/2} - \frac{x^4}{4} - \frac{x^6}{3X\,4^3} \right) dx$$

$$= k\left[2.\frac{x^{\frac{7}{2}}}{7/2} + \frac{8}{3}.\frac{x^{5/2}}{5/2} - \frac{1}{4}.\frac{x^5}{5} - \frac{1}{3X\,64}.\frac{x^7}{7} \right]_0^a$$

$$= k\left[\frac{2}{7}.256 + \frac{2}{15}.256 - \frac{256}{5} - \frac{256}{21} \right]$$

$$= \frac{6X\,256k}{35} = \frac{1536}{35}k$$

Q3)c) Solve $x^2 \frac{d^2 y}{dx^2} - 4x\frac{dy}{dx} + 6y = -x^4 \sin x$. **(8M)**

Ans : Putting z= log x and x =e^z , we get

$$[D(D-1) - 4D + 6)y = -e^{4z}.\sin e^z.$$

$$\therefore (D^2 - 5D + 6)y = -e^{4z}.\sin e^z$$

The Auxiliary Equation is

$$\therefore (D^2 - 5D + 6) = 0$$

$$\therefore (D-2)(D-3) = 0$$

$$\therefore D = 2,3$$

The Complementary Function is

$$y = c_1 e^{2z} + c_2 e^{3z}$$
$$P.I. = \frac{1}{D^2 - 5D + 6}(-e^{4z}\sin e^z)$$
$$= -e^{4z}.\frac{1}{(D+4)^2 - 5(D+4) + 6}\sin e^z$$
$$= -e^{4z}.\frac{1}{D^2 + 3D + 2}\sin e^z = -e^{4z}.\frac{1}{(D+2)(D+1)}\sin e^z$$
$$= -e^{4z}.\frac{1}{D+2}.e^{-z}.\int e^z \sin e^z dz$$

Put e^z=t

$$\therefore P.I = -e^{4z}.\frac{1}{D+2}.e^{-z}(-\cos e^z)$$
$$= e^{4z}.e^{-2z}.\int e^{2z}.e^{-z}\cos e^z dz \quad .$$
$$= e^{2z}\int e^z \cos e^z dz = e^{2z}.\sin e^z$$

Put $e^z = t$

The complete solution is

$$y = c_1 e^{2z} + c_2 e^{3z} + e^{2z}\sin e^z = c_1 x^2 + c_2 x^3 + x^2 \sin x \quad .$$

Q4)a) Find by double integration the area between the curves $y^2 = 4x$ and 2x-3y+4 = 0 . (6M)

Ans : We first solve the two equations to find the points of intersection .

We get

$$y^2 = 2(3y - 4)$$
$$y^2 - 6y + 8 = 0 \quad .$$
$$\therefore (y-4)(y-2) = 0, \therefore y = 2, 4$$

When y=2 , x=1 ;when y=4 , x=4 .Let the points of intersection be A(1,2) and B(4,4) .

Now, consider a strip parallel to the y -axis. On this strip, y varies from $y = \frac{2x+4}{3}$ to $y = 2\sqrt{x}$.

Then x varies from x=1 to x=4 .

$$\therefore A=\int_{1}^{4}\int_{(2x+4)/3}^{2\sqrt{x}} dydx=\int_{1}^{4}[y]_{(2x+4)/3}^{2\sqrt{x}}dx$$

$$=\int_{1}^{4}\left[2\sqrt{x}-\frac{(2x+4)}{3}\right]dx$$

$$=\left[2.\frac{2}{3}x^{3/2}-\frac{x^2+4x}{3}\right]_1^4$$

$$\therefore A=\left(\frac{32}{3}-\frac{32}{3}\right)-\left(\frac{4}{3}-\frac{5}{3}\right)=\frac{1}{3}$$

Q4)b) Solve $(1+\sin y)\frac{dx}{dy}=2y\cos y-x(\sec y+\tan y)$. **(6M)**

Ans : The given equation can be written as

$$\frac{dx}{dy}+\frac{(\sec y+\tan y)}{1+\sin y}.x=\frac{2y\cos y}{1+\sin y}$$

$$\therefore \frac{dx}{dy}+\sec y.\frac{(1+\sin y)}{(1+\sin y)}.x=\frac{2y\cos y}{1+\sin y}$$

$$\therefore \frac{dx}{dy}+\sec y.x=\frac{2y\cos y}{1+\sin y}.$$

$$\therefore e^{\int Pdy}=e^{\int \sec y}=e^{\log(\sec y+\tan y)}=\sec y+\tan y=\frac{1+\sin y}{\cos y}$$

Therefore, the solution is

$$x.\frac{(1+\sin y)}{\cos y}=\int\frac{2y\cos y}{(1+\sin y)}.\frac{(1+\sin y)}{\cos y}dy+c$$

$$=\int 2ydy+c=y^2+c.$$

The solution is $x.(1+\sin y)=y^2\cos y+c\cos y$

Q4)c) Solve $\frac{dy}{dx}=x^2+y^2$ with initial conditions $y_0=1$, $x_0=1$ at x=0.2 in steps of h=0.1 by Runge Kutta method of fourth order . (8M)

Ans : We have $\frac{dy}{dx}=x^2+y^2$

$$\therefore f(x,y)=x+y^2, x_0=0, y_0=1, h=0.1$$
$$k_1=hf(x_0,y_0)=0.1(0+1)=0.1$$
$$k_2=hf(x_0+\frac{h}{2},y_0+\frac{k_1}{2})=0.1\left[(0+0.05)+(1+0.05)^2\right]=0.1125$$
$$k_3=hf(x_0+\frac{h}{2},y_0+\frac{k_2}{2})=0.1\left[(0.05)+(1+0.05762)^2\right]=0.11686$$
$$k_4=hf(x_0+h,y_0+k_3)=0.1[(0+0.1]+(1+0.11686)^2]=0.13474$$
$$k=\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$
$$k=\frac{1}{6}[0.1+2(0.11525)+2(0.11686)+0.13474]=0.1165$$

The approximate value of y will be 1+0.1165 = 1.1165 .

Again to find at x = 0.2, we repeat the same .

$$\therefore f(x,y)=x+y^2, x=0.2, x_0=0.1, y_0=1.1165, h=0.1$$
$$k_1=hf(x_0,y_0)=0.1(0.1^2+1.1165^2)=0.13466$$
$$k_2=hf(x_0+\frac{h}{2},y_0+\frac{k_1}{2})=0.1\left[(0.1+0.05)+(1.1165+0.06733)^2\right]=0.15514$$
$$k_3=hf(x_0+\frac{h}{2},y_0+\frac{k_2}{2})=0.1\left[(0.1+0.05)+(1.1165+0.07757)^2\right]=0.15758$$
$$k_4=hf(x_0+h,y_0+k_3)=0.1[(0.1+0.1]+(1.1165+0.15758)^2]=0.18233$$
$$k=\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$
$$k=\frac{1}{6}[0.13466+2(0.15514)+2(0.15758)+0.18233]=0.1571$$

The approximate value of y = 1.1165 + 0.1571 = 1.2736 .

Q5)a) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx \,.\, \int_0^1 \frac{dx}{\sqrt{1-x^4}}$. **(6M)**

Ans : Put $x^4 = t$

$\therefore x = t^{1/4}$

$\therefore dx = \frac{1}{4}t^{-3/4}dt$ When x =0, t= 0 ; when x =1 , t =1

$$\therefore I = \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}}.\frac{1}{4}.t^{-3/4}dt.\int_0^1 \frac{1}{\sqrt{1-t}}.\frac{1}{4}.t^{-3/4}dt$$

$$= \int_0^1 \frac{1}{4}.t^{-1/4}(1-t)^{-1/2}dt.\int_0^1 \frac{1}{4}.t^{-3/4}(1-t)^{-1/2}dt$$

$$= \frac{1}{16}B\left(\frac{-1}{4}+1, \frac{-1}{2}+1\right).B\left(\frac{-3}{4}+1, \frac{-1}{2}+1\right)$$

$$= \frac{1}{16}B\left(\frac{3}{4}, \frac{1}{2}\right).B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{16}.\frac{\neg(3/4)\neg(1/2)}{\neg(5/4)}.\frac{\neg(1/4)\neg(1/2)}{\neg(3/4)}$$

$$= \frac{1}{16}.\frac{\sqrt{\pi}}{1/4\neg(1/4)}.\neg(1/4).\neg(1/2)$$

$$= \frac{1}{4}\left(\sqrt{\pi}\right)^2 = \frac{\pi}{4}$$

Q5)b) The distance x descended by a parachute satisfies the differential equation
$\left(\frac{dx}{dt}\right)^2 = k^2\left(1-e^{-2gx/k^2}\right)$ **where k and g are constants. If x=0 when t=0, show that**
$x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$ **.** **(6M)**

Ans : We have

$$\frac{dx}{dt} = k\sqrt{1-e^{-2gx/k^2}}$$

$$\therefore \frac{dx}{\sqrt{1-e^{-2gx/k^2}}} = kdt$$

Let $\sqrt{1-e^{-2gx/k^2}}=u$.

$\therefore 1-e^{-2gx/k^2}=u^2$.

$\therefore e^{-2gx/k^2}.\frac{g}{k^2}.dx=udu$

$\therefore (1-u^2)\frac{g}{k^2}dx=udu$

$\therefore dx=\frac{k^2}{g}.\frac{u}{1-u^2}du$

Hence we get that

$$\frac{k^2}{g}.\frac{u}{1-u^2}.\frac{1}{u}du=kdt$$

.

$$\therefore \frac{k}{g}.\frac{du}{1-u^2}=dt+c$$

By integration, $\frac{k}{g}.\frac{1}{2}\log\left(\frac{1+u}{1-u}\right)=t+c$.

But

$$\frac{1}{2}\log\left(\frac{1+u}{1-u}\right)=\tanh^{-1}u$$

.

$$\therefore \frac{k}{g}\tanh^{-1}u=t+c$$

But by data when t=0, x=0 and hence u=0 .Therefore , c=0 .

$$\therefore \frac{k}{g}\tanh^{-1}u=t$$

$$\tanh^{-1}u=\frac{gt}{k}$$

$$u=\tanh\frac{gt}{k}$$

.

$$\therefore u^2=\tanh^2\left(\frac{gt}{k}\right)$$

$$\therefore 1-e^{-2gx/k^2}=\tanh^2\left(\frac{gt}{k}\right)$$

$$e^{-2gx/k^2}=1-\tanh^2\left(\frac{gt}{k}\right)=\sec h^2\left(\frac{gt}{k}\right)$$

$$e^{2gx/k^2} = \cos h^2\left(\frac{gt}{k}\right)$$

$$\therefore \frac{2gx}{k^2} = 2\log\cosh\left(\frac{gt}{k}\right)$$

$$\therefore x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$$

Q5)c) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using i) Trapezoidal ii) Simpsons (1/3)rd and iii) Simpsons (3/8)th rule . (8M)

Ans : Firstly, we shall divide the interval (0,1) into 10 equal parts by taking h=0.1. We prepare the following table :

x :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y :	1	0.9901	0.9615	0.9174	0.8621	0.8	0.7353	0.6711	0.6098	0.5525	0.5

(i) By Trapezoidal Rule

$$I = \frac{h}{2}[X + 2R]$$

X = 1.5 , R =7.0998

$$I = \frac{0.1}{2}[1.5 + 2X\,7.0998]$$

$$\therefore I = 0.7849$$

(ii) By Simpson's (1/3)rd rule

$$S = \frac{h}{3}[X + 2E + 4O] .$$

X = 1.5 , E = 3.1687 , O = 3.9311

$$S = \frac{0.1}{3}[1.5 + 2X\,3.1687 + 4X\,3.9311]$$

S = 0.7853

(iii) By Simpson's (3/8)th rule

$$S = \frac{3h}{8}[X + 2T + 3R]$$

X = 1.5 , T= 2.2052 , R= 4.8946

$$S = \frac{3X\,0.1}{8}[1.5 + 2X\,2.2052 + 3X\,4.8946]$$

S = 0.7834

Q6)a) Find the volume in the first octant bounded by the cylinder $x^2+y^2=2$ and the planes $z=x+y, y=x, z=0$ and x=0 . **(6M)**

Ans : If we take projections on the xy – plane ,the area is bounded by the circle $x^2+y^2=2$, the line y=x and the line x=0 i.e. the y-axis .

We change the co-ordinates to cylindrical polar by putting x=rcosΘ , y=rsinΘ , z=z .

Then the equation of the cylinder becomes $x^2+y^2=2$ i.e. $r=\sqrt{2}$.

The line y = x becomes , $r\sin\theta = r\cos\theta, \therefore \theta=\frac{\pi}{4}$.

The line x=0 becomes, $r\cos\theta=0, \therefore \theta=\frac{\pi}{2}$.

Now if we consider a radial strip in the projection , r varies from r=0 to $r=\sqrt{2}$, Θ varies from $\theta=\frac{\pi}{4}$ to $\theta=\frac{\pi}{2}$. Then z varies from z = 0 to z = (x+y) = r(cosΘ+sinΘ) .

$$\therefore V = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}}\int_{z=0}^{r(\cos\theta+\sin\theta)} r\,dr\,d\theta\,dz = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r[z]_0^{r(\cos\theta+\sin\theta)}\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\sqrt{2}} r^2(\cos\theta+\sin\theta)\,dr\,d\theta = \int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$$

$$= \frac{2\sqrt{2}}{3}\int_{\theta=\pi/4}^{\pi/2}(\cos\theta+\sin\theta)\,d\theta = \frac{2\sqrt{2}}{3}[\sin\theta-\cos\theta]_{\pi/4}^{\pi/2}$$

$$= \frac{2\sqrt{2}}{3}\left[(1-0)-\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\right] = \frac{2\sqrt{2}}{3}$$

Q6)b) Change the polar coordinates and evaluate $\iint_R \frac{dxdy}{(1+x^2+y^2)^2}$ over one loop of the lemniscates $(x^2+y^2)^2 = x^2-y^2$. **(6M)**

Ans : If we put $x=2\cos\theta$ and $y=r\sin\theta$, $(x^2+y^2)^2=x^2-y^2$ becomes

$r^4 = r^2(\cos^2\theta-\sin^2\theta) i.e.$

$r^2=\cos 2\theta$

$$\therefore \frac{1}{(1+x^2+y^2)^2} = \frac{1}{(1+r^2)^2}$$

Now, the loop varies from 0 to $\sqrt{\cos 2\theta}$ and Θ varies from $-\pi/4$ to $\pi/4$.

$$\therefore I=\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}\frac{r\,dr\,d\theta}{(1+r^2)^2}$$

$$=2\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}\frac{r\,dr\,d\theta}{(1+r^2)^2}$$

$$=-\int_0^{\pi/4}\left[\frac{1}{1+\cos 2\theta}-1\right]d\theta=\int_0^{\pi/4}\left[1-\frac{1}{1+\cos 2\theta}\right]d\theta$$

$$=\int_0^{\pi/4}\left[1-\frac{\sec^2\theta}{2}\right]d\theta=\left[\theta-\frac{\tan\theta}{2}\right]_0^{\pi/4}=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$

Q6)c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}\ .$$ **(8M)**

Ans : The Auxiliary equation is

$$D^2-1=0$$
$$\therefore (D-1)(D+1)=0\ .$$
$$\therefore D=1,-1$$

The Complementary Function is $y=c_1e^x+c_2e^{-x}$.

Hence, $y_1=e^x, y_2=e^{-x}$ and $X=\dfrac{2}{1+e^x}$.

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-2e^0=-2$$

Now, $\therefore u=-\int\frac{y_2X}{W}dx=-\int\frac{e^{-x}}{2}.\frac{2}{1+e^x}dx=\int\frac{e^{-x}}{1+e^x}dx$.

Put $e^{-x}=t$

$e^{-x}dx=-dt$

$$\therefore u=-\int\frac{y_2X}{W}dx=-\int\frac{e^{-x}}{2}.\frac{2}{1+e^x}dx=\int\frac{e^{-x}}{1+e^x}dx\ .$$

$$e^{-x}dx = -dt$$

$$\therefore u = -\int \frac{dt}{1+(1/t)} = -\int \frac{t}{1+t}dt$$

$$= -\int \frac{(t+1)-1}{t+1}dt = -\int 1dt + \int \frac{dt}{1+t}$$

$$= -t + \log(1+t) = -e^{-x} + \log(1+e^{-x})$$

$$v = \int \frac{y_1 X}{W}dx = \int \frac{e^x}{-2}.\frac{2}{1+e^x}dx = -\int \frac{e^x}{1+e^x}dx = -\log(1+e^x)$$

$$\therefore P.I = uy_1 + vy_2 = \left[-e^{-x} + \log(1+e^{-x})\right]e^x + \left[-\log(1+e^x)\right]e^{-x}$$

The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - 1 + e^x.\log(1+e^{-x}) - e^{-x}.\log(1+e^x) \quad .$$
