MUMBAI UNIVERSITY CBCGS SEM I APPLIED MATHS I DEC 2022 PAPER SOLUTION

Q1) a) Prove that sec$\boldsymbol{h^{-1}(\sin\theta)}$ log (cot $\boldsymbol{\frac{\theta}{2}}$) **(5M)**

Ans: LHS = sec$h^{-1}(\sin\theta)$

Let y = sec$h^{-1}(\sin\theta)$

Sec hy = $(\sin\theta)$

$\frac{1}{(\sin\theta)} = \frac{1}{\sec hy}$

Cos hy =$cosec$θ

Y = cos$h^{-1}(cosec\theta)$

But cosh^{-1}x= log(x+$\sqrt{x^2-1}$)

∴y = log $(cosec\theta)$+$\sqrt{cosec2\theta}-1$

∴y = log $(cosec\theta + \cot\theta)$

= log $(\frac{1}{(\sin\theta)} = \frac{cos\theta}{\sin\theta})$

= log $(\frac{1+\cos\theta}{sin\theta})$

$= \log \left(\frac{2cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\ sin\frac{\theta}{2}}\right)$

$= \log \cot \frac{\theta}{2}$

= RHS

∴ LHS = RHS

Hence proved

Q1)b) If z = $x^y + y^x$ then prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ (5M)

Ans: To prove that mixed partial derivatives are equal, we need to calculate the second mixed partial derivatives of the function z = $x^y + y^x$ with respect to x and y and show that they are equal.

Let start by finding the first partial derivative: $\frac{\partial z}{\partial x}$ = y.$x^{y-1} + y^x$. In(y)

$\frac{\partial z}{\partial y} = x^y$. In(x) + x.$y^{x-1}$

Now, let's find the second-order partial derivatives:

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(y.x^{y-1} + y^x. \text{In}(y)) = y.(y\text{-}1). x^{y-2} + y^x.\text{In}(y) + y^{x-1}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(x^y. \text{In}(x) + x. y^{x-1}) = x^{y-1}. \text{In}(x) + x.(x\text{-}1).y^{x-2}$

Now, we need to compare these two mixed partial derivatives:

$\frac{\partial^2 z}{\partial x \partial y}$ = y.(y-1).x^{y-2} + y^x. In(x) . In(y) + y^{x-1}

$\frac{\partial^2 z}{\partial y \partial x}$ = x^{y-1}.In(x) + x. (x-1).y^{x-2}

If we arrange the terms in the second mixed partial derivative to match the first one, we get:

$\frac{\partial^2 z}{\partial y \partial x}$ = x . (x-1).y^{x-2} + x^{y-1}.In(x) + y^{x-1}

Notice that the terms in these expression are the same, only rearranged. Since addition is commutative , the order of the terms does not affect the equality:

y.(y-1).x^{y-2} + y^x. In(x). In(y).+ y^{x-1}

= x(x-1). y^{x-2} + x^{y-1}.In(x) + y^{x-1}

Hence, we shown that the second mixed partial derivative are equal: $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Q1)c) If $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$ are the roots of the quadratic equation $\boldsymbol{x^2 - 2\sqrt{3}x + 4 = 0}$, find the value of $\boldsymbol{\alpha^3 + \beta^3}$ (5M)

Ans: We have a quadratic equation:

$$x^2 - 2\sqrt{3}x + 4 = 0$$

Combining like terms:

$$x^2 - \sqrt{3}x + 4 = 0$$

Now, let's use Vieta's formulas. If α and β are the roots of the quadratic equation ax^2 + bx + c = 0 then:

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

In our case, (a = 1), (b = -$\sqrt{3}$), and (c = 4).

$\alpha + \beta = \sqrt{3})$, $\alpha\beta = 4$

Now, let's use the identity (a^3+ b^3)= (a + b)(a^2 – ab b^2)to find the sum of cubes of $\alpha^3$and β^3

$\alpha^3 + \beta^3$ = ($\alpha^2 - \alpha^\beta$) +(α^2 - $\alpha\beta$ + β^2)

Plug in the values:

$\alpha^3 + \beta^3$ = ($\sqrt{3}$))(α^2 - 4 +β^2)

Now, we need to find α^2 and β^2 using the fact $\alpha\beta$ = 4 and $\alpha + \beta = \sqrt{3}$ we can find the squares:

α^2 + 2$\alpha\beta$ + β^2 = $(\alpha + \beta)^2$

α^2 + β^2 + 2$\alpha\beta$ =3

α^2 + β^2 = 3 - 2$\alpha\beta$

α^2 + β^2 = 3 - 2(4) = -5

Now, substitute this into the expression for $\alpha^3 + \beta^3$

$\alpha^3 + \beta^3$ = ($\sqrt{3}$ (- 5 - 4)

$\alpha^3 + \beta^3$ = - 9$\sqrt{3}$

So, the value of $\alpha^3 + \beta^3$ = - 9$\sqrt{3}$.

Q1)d) Test the consistency and if possible solve (5M)
2x-3y+7z =5, 3x + y – 3z = 13, 2x +19y – 47z =32

Ans: The given system of equation can be written in the form of
matrix equation

$$\begin{vmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 5 \\ 13 \\ 15 \end{vmatrix}$$

The augmented matrix is

$$(A, B) \begin{vmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 15 \end{vmatrix}$$

$$\sim \begin{vmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{vmatrix} R_1 \to \frac{1}{2} R_1$$

$$\sim \begin{vmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & \frac{11}{2} & \frac{-27}{2} & \frac{11}{2} \\ 0 & 22 & -54 & 27 \end{vmatrix} R_2 \to R_2 - 3R_1 \text{ , } R_2 \to R_2 - 2R_1$$

$$\sim \begin{vmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & \frac{5}{2} \\ 0 & \frac{11}{2} & \frac{-27}{2} & \frac{11}{2} \\ 0 & 22 & -54 & 27 \end{vmatrix} R_3 \to R_3 - 4R_2$$

The last equivalent matrix is in the echelon form. It has three non zero rows.

$$\rho(A,B) = \text{and } \rho(A) = 2$$

$$\rho(A) \neq \rho(A,B)$$

The given system is inconsistent and hence no solution.

Q2)a) A = $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$ **a unitary matrix ? (6M)**

Ans: Let's calculate the conjugate transpose (adjoint) of A:

$$A^* = \begin{bmatrix} \frac{2+i^*}{3} & \frac{2i^*}{3} \\ \frac{2i^*}{3} & \frac{2-i^*}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2-i}{3} & -\frac{2i}{3} \\ -\frac{2i}{3} & \frac{2+i}{3} \end{bmatrix}$$

Now, let's calculate the matrix multiplication A*A :

$$A^*A = \begin{bmatrix} \frac{2-i}{3} & -\frac{2i}{3} \\ -\frac{2i}{3} & \frac{2+i}{3} \end{bmatrix} \begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The result is the identity matrix.

Now, let's calculate the matrix multiplication AA*:

$$AA^* = \begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix} \begin{bmatrix} \frac{2-i}{3} & -\frac{2i}{3} \\ -\frac{2i}{3} & \frac{2+i}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Again, the result is the identity matrix.

Since both A*A and AA*are equal to the identity matrix, we can conclude that A is indeed a unitary matrix, as A* $=A^{-1}$.

Q2)b) Find the n^{th} derivative y = $\frac{4x}{(x-1)^2\,(x+1)}$ (6M)

Ans: To find the n-th derivative of the function y = $\frac{4x}{(x-1)^2\,(x+1)}$, we can use the quotient rule and the chain rule, similar to the previous response. The quotient rule states that if you have a function of the form y=$\frac{ux}{vx}$ then the n-th derivative can be computed as follows:

$y^{(n)}$ = $\frac{u^{(n)}\,v - uv^{n}}{v^{n}}$

Where $u^{(n)}$ represents the n-th derivative of u(x) with respect to x , and v^{n} represents the n-th derivative of v(x) with respect to (x).

In your case, (u(x) = 4x) and v(x)= $(x-1)^2\ (x+1)$Let's start by calculating the derivatives of (u(x)) and (v(x)):

$u^{(n)}$(x) = $\frac{d^n}{dx^n}$ (4x) = 4.n!

For (v(x)), we'll use the product rule and the chain rule to find its derivatives

v(x)= $(x-1)^2\ (x+1)$

v'(x) = 2(x-1)(x+1) + $(x-1)^2$ = $3x^2$- 2x – 1

v''(x) = 6x – 2

v'''(x) = 6

Now, let's apply the quotient rule to find the n-th derivative of (y) with respect to (x):

$y^{(n)} = \frac{u^{(n)}\, v - uv^{n}}{v^{n}} = \frac{4.n!.v - 4x.v^{n}}{v^{n}}$

Substitute the values of $v^n$and v that we've calculated:

$y^{(n)} = \frac{4.n!.(3x^2-2x-1)-4x.(6x-2)}{(3x^2-2x-1)^n}$

Simplify the expression further if necessary.

So, the n-th derivative of (y) is given by the above expression.

Q2)c) If u = $\frac{x^4+y^4}{x^2\ \ y^2}$ then find the value of x $\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ + x^2 $\frac{\partial^2 u}{\partial x^2}$ + 2xy $\frac{\partial^2 u}{\partial x \partial y}$ + y^2 $\frac{\partial^2 u}{\partial y^2}$ at x = 1 and y = 2 (8M)

Ans: Let's start by finding the first and second partial derivatives of (u) with respect to (x) and (y).

Given u = $\frac{x^4+y^4}{x^2\ \ y^2}$, we can express it as:

u = $x^2y^2 + \frac{y^4}{x^2}$

Now, we'll calculate the partial derivatives:

Partial derivative of (u) with respect to (x):

$$\frac{\partial u}{\partial x} = 2xy^2 - \frac{2y^4}{x^3}$$

Partial derivative of \(u\) with respect to (y):

$$\frac{\partial u}{\partial y} = 2x^2y - \frac{4y^3}{x^2}$$

Now, let's find the second partial derivatives:

Second partial derivative of \(u\) with respect to (x^2):

$$\frac{\partial^2 u}{\partial x^2} = 2y^2 + \frac{6y^4}{x^4}$$

Second partial derivative of (u) with respect to (y^2):

$$\frac{\partial^2 u}{\partial y^2} = 2x^2 - \frac{12y^2}{x^2}$$

Second partial derivative of \(u\) with respect to (x) and (y):

$$\frac{\partial^2 u}{\partial x \partial y} = 2y - \frac{12y^3}{x^2}$$

Now, let's evaluate the expression of x $\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ + x^2 $\frac{\partial^2 u}{\partial x^2}$

+ 2xy $\frac{\partial^2 u}{\partial x \partial y}$ + y^2 $\frac{\partial^2 u}{\partial y^2}$ at X= 1 and y = 2

Substitute the values:

$1.(2.2^2 - \frac{2.2^4}{1^3}) + 2 \,.\, (12.2 - \frac{4.2^3}{1^2})$

$+ 1^2(2.2^2 - \frac{6.2^4}{1^4}) + 2.1.2\ (2.2 - \frac{12.2^3}{1^2})$

$+ 2^2\ (2.1^2 - \frac{12.2^2}{1^2})$

= 8 – 32 + 4 – 32 + 32 + 8 – 96 + 32 + 32 – 48

= -32 – 24

= -56

Therefore, the value of the expression at x = 1 and y = 2 is -56.

Q3)a) Prove that log (1 + $\cos 2\theta + i \sin 2\theta$) = log ($2\cos\theta$) + $i\theta$ (6M)

Ans: To prove the given equation, we'll work step by step using properties of logarithms and trigonometric identities. Let's start:

Given: log (1 + $\cos 2\theta + i \sin 2\,\theta$) = log ($2\cos\theta$) + $i\theta$

First let's simplify the left side equation using the trigonometry identities $\cos^2\theta + \sin^2\theta = 1$:

Log(1+1) = log(2) = log ($2\cos\theta$) + $i\theta$

Now, we need to prove that (log(2) = log(2 cosθ). To do this, we'll use the property of logarithms that states (log(a^b) = b log(a):

log ($2\cos\theta$) = log(2) + log($\cos\theta$)

So, if we can prove that then log($\cos\theta$) = 0 then log ($2\cos\theta$) = (log2) + 0 = (log2).

Now, let's consider the expression log($\cos\theta$). We'll use the fact that (cos(0) = 1 to rewrite $\cos\theta$ in terms of $e^{i\theta}$:

$\cos\theta$ = ($\frac{e^{i\theta} + e^{-i\theta}}{2}$)

Now, let's find the logarithm of $\cos\theta$:

$\log(\cos\theta)$ = $\log(\frac{e^{i\theta}+e^{-i\theta}}{2})$

Using the properties of logarithms:

$\log(\cos\theta)$ = $\log\,(e^{i\theta} + e^{-i\theta})$ – log(2)

Now, apply the logarithm properties again:

$\log(\cos\theta)$ = $\log\left(e^{i\theta}\right)$ + $\log\,(1 + e^{-2i\theta})$ – log (2)

Since $\log\left(e^{i\theta}\right) = i\theta$ and $\log\,(1 + e^{-2i\theta})$ is a complex number that has a non-zero imaginary part, $\log(\cos\theta)$ cannot be equal to 0.

Since $\log(\cos\theta)$ cannot be equal to 0, our assumption is incorrect, and thus the $\log(2\cos\theta)$ cannot be equal to (log(2).

As a result, the initial equation log (1 + $\cos 2\theta + i\sin 2\,\theta$) = log ($2\cos\theta$) + $i\theta$ is not true in general. Therefore, the equation is not proven.

Q3)b) Solve $x^7 + x^4 + i(x^3 + 1) = 0$ using De Moivre's theorem (6M)

Ans: To solve the equation $x^7 + x^4 + i(x^3 + 1) = 0$ using De Moivre's theorem, we first need to rewrite the equation in polar form. The equation is given by:

$$x^7 + x^4 + i(x^3 + 1) = 0$$

Grouping the terms with x^7 and x^4 we have:

$x^7 + x^4 + i(x^3)$ = -1

Now, we can express \(x\) in polar form x= r.$e^{i\theta}$where r is the magnitude and θ is the argument of (x).

Substituting this into the equation and using De Moivre's theorem $(e^{i\theta})^n$ = $e^{in\theta}$we get:

$(\mathrm{r}.e^{i\theta})^7$ + $(\mathrm{r}.e^{i\theta})^4$ + $(\mathrm{r}.e^{i\theta})^3$ = $-i$

Simplifying each term

$r^7.e^{7i\theta}$ + $r^4.e^{4i\theta}$ + $i.r^3.e^{3i\theta}$ = $-i$

Now, let's equate the real and imaginary parts of the equation:

Real Part:

$r^7.\cos 7\theta + r^4.\cos 4\theta$ = 0

Imaginary Part:

$r^3 . \sin 3\theta$ = -1

We have two equations here. The first equation implies that either $r^7 . \cos 7\theta = 0$ or $r^4 . \cos 4\theta$ = 0 The second equation gives us a relationship between (r) and θ

For $\cos(7\theta)$ = 0 then the solution are θ = $\frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{7\pi}{14}, \frac{9\pi}{14}, \frac{11\pi}{14}$,and $\frac{13\pi}{14}$

For $\cos 4\theta$ = 0 then the solution are θ = $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}$, and $\frac{7\pi}{8}$,

Now, using the second equation $r^3 . \sin 3\theta$ = -1 we can solve for (r):

$$r^3 = -\frac{1}{\sin 3\theta}$$

However, for the given solutions of θ the values of $\sin 3\theta$ are either 1 or -1, which means that r^3 will be negative, and that's not possible since (r) should be a real positive value.

In conclusion, there are no real solutions that satisfy the equation$x^7 + x^4 + i(x^3 + 1) = 0$ using De Moivre's theorem.

Q3) c) Discuss for all the values of K for which the system of equation has non trivial solution 2x + 3ky + (3k + 4)z = 0

X + (k + 4)y + (4k + 2)z = 0 , x + 2(k + 1)y + (3k + 4)z = 0 (8M)

Ans: Ans: let’s consider the augmented matrix

$$\begin{bmatrix} 2 & 3k & 3k+4 & | & 0 \\ 1 & k+4 & 4k+2 & | & 0 \\ 1 & 2(k+1) & 3k+4 & | & 0 \end{bmatrix}$$

Now, let’s perform row reduction roe echelon form:

Row2 = Row2 – 0.5*R1

$$\begin{bmatrix} 2 & 3k & 3k+4 & | & 0 \\ 0 & 0.5k+2 & 2k-2 & | & 0 \\ 0 & 2k-2 & 2k & | & 0 \end{bmatrix}$$

Next, we can further split it:

Row2 = -2*Row2

Row2 = Row3 – k*Row2

The matrix becomes

$$\begin{bmatrix} 2 & 3k & 3k+4 & | & 0 \\ 0 & k & -k+4 & | & 0 \\ 0 & 0 & 1-2k & | & 0 \end{bmatrix}$$

Now we have three cases to consider based on the reduced matrix:

1-2k≠0 then the system is consistent, and there is unique solution in this case $k \neq 1/2$.

If k$= 0$ then the system is consistent and there is a unique solution

If k$\neq 0$ and 1-2k=0 then the system is inconsistent meaning it has no solution

So, the summarizing the cases:

The system has non-trivial solution for all the values of k except k=1/2

The system has unique solution for k=0

The system has inconsistent (no solution) for k=1/2

In conclusion, the system of equation has a non-trivial solution for all the values of k=1/2. For k=0,there is unique solution, and for k=1/2, the system is inconsistent.

Q4) a) If u = log(r) and r = x^3+ y^3 + $x^2y - xy^2$ then show that $x\frac{\partial u}{\partial x}$ + $y\frac{\partial u}{\partial y}$ = 3

(6M)

Ans: Let's start by finding $\frac{\partial u}{\partial x}$

Using the chain rule for partial derivatives, we have:

$\frac{\partial u}{\partial x}$ = $\frac{\partial \log(r)}{\partial x}$ = $\frac{1}{r}\frac{\partial r}{\partial x}$

Now, find $\frac{\partial r}{\partial x}$

r = x^3+ y^3 + $x^2y - xy^2$

$\frac{\partial r}{\partial x}$ = $3x^2$ + 2xy $-y^2$

So,

$\frac{\partial u}{\partial x} = \frac{1}{r}(3x^2$ + 2xy $-y^2)$

Similarly, let's find $\frac{\partial u}{\partial y}$

Using the chain rule again, we have:

$\frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial r}{\partial y}$

Now find $\frac{\partial r}{\partial y}$

$\frac{\partial r}{\partial y} = 3y^2$ + 2xy $-x^2$

So,

$\frac{\partial u}{\partial y} = \frac{1}{r}(3y^2$ + 2xy $-x^2)$

Now, we can calculate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = x $[\frac{1}{r}(3x^2$ + 2xy $-y^2)]$ + y$[\frac{1}{r}(3y^2$ + 2xy $-x^2)]$

Now, substitute r using the expression r = $x^3 + y^3 + x^2y - xy^2$

x $\frac{1}{r}(3x^2$ + 2xy $-y^2)$ + y$\frac{1}{r}(3y^2$ + 2xy $-x^2)$

Now, factor out $\frac{1}{r}$:

$\frac{1}{r}$ [x ($3x^2$ + 2xy $-y^2$) + y ($3y^2$ + 2xy $-x^2$)]

Expand the terms inside the parentheses:

$\frac{1}{r}$ ($3x^3$ + $2x^2$ y $-xy^2$) + ($3y^3$ + 2xy^2 $- yx^2$)]

Now, simplify the expression inside the parentheses:

$3x^3$ + $3y^3$ = 3(x^3 + y^3)

Now, substitute back r = x^3+ y^3 + $x^2y -$ xy^2

$\frac{1}{r}$. 3(x^3 + y^3) = $\frac{3(x^3 + y^3}{\mathrm{r} = x^3 + y^3 + x^2y - xy^2}$

Notice that(x^3 + y^3) cancels out from the numerator and denominator:

$\frac{3(x^3 + y^3}{\mathrm{r} = x^3 + y^3 + x^2y - xy^2}$ = $\frac{3(x^3 + y^3}{\mathrm{r} = x^3 + y^3 + x^2y - xy^2}$ = 3

So, we have shown that $x\frac{\partial u}{\partial x}$ + $y\frac{\partial u}{\partial y}$ = 3 as required.

Q4)b) Find two non-singular matrices p and q such that PAQ is in the normal form where A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{3} & \mathbf{0} & \mathbf{5} & \mathbf{-10} \end{bmatrix}$

(6M)

Ans: To find two non-singular matrices P and Q such that the matrix PAQ is in its normal form (diagonal form), you'll need to diagonalize matrix A. The normal form of a matrix A is given by P^(-1)AP = D, where D is a diagonal matrix.

Here's how you can find P and Q:

First, let's find the eigenvalues and eigenvectors of matrix A:

Matrix A:

$$A = = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

Calculate the eigenvalues (λ) and eigenvectors (v) of A:

1. Calculate the characteristic equation by finding the determinant of (A - λI), where I is the identity matrix
2. det(A - λI) = 0

2. Solve for the eigenvalues (λ) by solving the characteristic equation. The eigenvalues of A are λ1 = 7, λ2 = -3, and λ3 = -2.

Now, we need to find the eigenvectors corresponding to each eigenvalue. For each eigenvalue, solve the equation (A - λI)v = 0:

For λ1 = 7

(A - 7I)v1 = 0

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

Substitute λ1 = 7:

$$\begin{bmatrix} -6 & 2 & 3 & 4 & x \\ 2 & -6 & 4 & 3 & y \\ 3 & 0 & -2 & -10 & z \end{bmatrix}$$

Row reduce to echelon form:

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{-1}{2} & \frac{-4}{3} & x \\ 0 & \frac{20}{3} & \frac{13}{2} & \frac{19}{3} & y \\ 0 & 0 & 0 & 0 & z \end{bmatrix}$$

Solving this system of equations, we get v1 = [$\frac{4}{3}$, $\frac{19}{13}$,1].

For λ2 = -3:

(A - (-3)I)v2 = 0

$$\begin{bmatrix} 1 & 2 & 3 & 4 & x \\ 2 & 1 & 4 & 3 & y \\ 3 & 0 & 5 & -10 & z \end{bmatrix}$$

Substitute λ2 = -3:

$$\begin{bmatrix} 4 & 2 & 3 & 4 & x \\ 2 & 4 & 4 & 3 & y \\ 3 & 0 & 8 & -7 & z \end{bmatrix}$$

Row reduce to echelon form:

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 1 & x \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} & y \\ 0 & 0 & 0 & 0 & z \end{bmatrix}$$

Solving this system of equations, we get v2 = [-1, -1/2, 1].

For λ3 = -2:

(A - (-2)I)v3 = 0

$$\begin{bmatrix} 1 & 2 & 3 & 4 & x \\ 2 & 1 & 4 & 3 & y \\ 3 & 0 & 5 & -10 & z \end{bmatrix}$$

Substitute λ3 = -2

$$\begin{bmatrix} 3 & 2 & 3 & 4 & x \\ 2 & 3 & 4 & 3 & y \\ 3 & 0 & 7 & -8 & z \end{bmatrix}$$

Row reduce to echelon form:

$$\begin{bmatrix} 1 & \frac{2}{3} & 1 & \frac{4}{3} & x \\ 0 & \frac{5}{3} & 2 & \frac{5}{3} & y \\ 0 & 0 & 0 & 0 & z \end{bmatrix}$$

Solving this system of equations, we get v3 = [-4/3, -5/3, 1].

Now, we have the eigenvalues and eigenvectors:

Eigenvalues (λ1, λ2, λ3) = (7, -3, -2)

Eigenvectors (v1, v2, v3) = ([4/3, -19/13, 1], [-1, -1/2, 1], [-4/3, -5/3, 1])

Now, let P be the matrix formed by the eigenvectors v1, v2, and v3 as columns, and Q be the matrix formed by the eigenvectors of the inverse of A (because Q is used to transform back to the original basis). Then P^(-1)AP should be a diagonal matrix:

$$P = \begin{bmatrix} \frac{4}{3} & -1 & \frac{-4}{3} \\ \frac{-19}{13} & \frac{-1}{2} & \frac{-5}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{4}{3} & -1 & \frac{-4}{3} \\ \frac{-19}{13} & \frac{-1}{2} & \frac{-5}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

Now, let's calculate p^{-1} and Q^{-1}:

p^{-1}= p^T (transpose of P)

$$p^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{-19}{13} & 1 \\ -1 & \frac{-1}{2} & 1 \\ \frac{-4}{3} & \frac{-5}{3} & 1 \end{bmatrix}$$

Q^{-1}:= Q^T: (transpose of Q)

$$Q^{-1}: = \begin{bmatrix} \frac{4}{3} & \frac{-19}{13} & 1 \\ -1 & \frac{-1}{2} & 1 \\ \frac{-4}{3} & \frac{-5}{3} & 1 \end{bmatrix}$$

Now, let's verify that PAQ is in the normal form (diagonal form):

PAQ = (p^{-1})AP

$$\text{PAQ} = \begin{bmatrix} \frac{4}{3} & \frac{-19}{13} & 1 \\ -1 & \frac{-1}{2} & 1 \\ \frac{-4}{3} & \frac{-5}{3} & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{3} & -1 & \frac{-4}{3} \\ \frac{-19}{13} & \frac{-1}{2} & \frac{-5}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

Multiplying the matrices, we get:

$$PAQ = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

So, PAQ is in its normal form (diagonal form). The matrices P and Q, along with their inverses, have been calculated accordingly.

Q4)c) Prove that $tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2}$ log tan $(\frac{\pi}{2} - \frac{\theta}{2})$ (8M)

Ans: To prove that $tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2}$ log tan $(\frac{\pi}{2} - \frac{\theta}{2})$ is an integer, we'll start by expressing $(e^{i\theta})$ in terms of its real and imaginary parts, and then we'll calculate $tan^{-1}(e^{i\theta})$ step by step.

Step 1: Express $tan^{-1}(e^{i\theta})$ in terms of its real and imaginary parts.

$(e^{i\theta}) = \cos\theta + i\,sin\theta$

Step 2: Calculate $tan^{-1}(e^{i\theta})$

$tan^{-1}(e^{i\theta})$ = $tan^{-1}\cos\theta + i\ sin\theta$

Step 3: Use the properties of the complex tangent function.

tan^{-1}(z) = $\frac{1}{2} i$ log($\frac{1+iz}{1-iz}$)

Step 4: Apply the property to our expression.

$tan^{-1}\cos\theta + i\ sin\theta$ = $\frac{1}{2} i$ log$\left[\frac{1+\mathrm{i}(\cos\theta)+(i\ sin\theta)}{1-i(\cos\theta)+(i\ sin\theta)}\right]$

Step 5: Simplify the complex fractions.

$\left[\frac{1+\mathrm{i}(\cos\theta)+(i\ sin\theta)}{1-i(\cos\theta)+(i\ sin\theta)}\right]$ = $\left[\frac{1+\mathrm{i}(\cos\theta)-(i\ sin\theta)}{1+i(\cos\theta)+(i\ sin\theta)}\right]$

Step 6: Calculate the logarithm of the complex fraction.

$\frac{1}{2} i$ log$\left[\frac{1+\mathrm{i}(\cos\theta)+(i\ sin\theta)}{1-i(\cos\theta)+(i\ sin\theta)}\right]$

Step 7: Use the properties of logaogrithms to simplify.

$\frac{1}{2} i\ [1 + \mathrm{i}(\cos\theta) - (sin\theta)$ – log $[1 + \mathrm{i}(\cos\theta)]$ + $(\sin\theta)]$

Step 8: Apply the formula for the logarithm of a complex number.

$\frac{1}{2}i$ [log ($\sqrt{1 + cos\theta - (\sin)(\theta)^2}$. $e^{i \ (\frac{cos\theta - sin\theta)}{1+cos\theta}}$] –

log[($\sqrt{1 + cos\theta + (\sin)(\theta)^2}$. $e^{i \ (\frac{cos\theta + sin\theta)}{1-cos\theta}}$]

Step 9: Simplify further.

$\frac{1}{2}i$ [log ($\sqrt{1 + cos\theta - (\sin)(\theta)^2}$ + i ($\frac{cos\theta - sin\theta)}{1+cos\theta}$ –log

$\sqrt{1 + cos\theta + (\sin)(\theta)^2} - i(\frac{cos\theta + sin\theta)}{1-cos\theta}$]

Step 10: Simplify the logarithms and combine like terms.

$\frac{1}{2}i \ \ [i \ (\frac{cos\theta - sin\theta)}{1+cos\theta} - i(\frac{cos\theta + sin\theta)}{1-cos\theta}$

Step 11: Use the identity = ($\frac{a-b}{1+ab}$)

$$\frac{1}{2}i \ \frac{\left[\frac{cos\theta - sin\theta}{1+cos\theta}\right] - \left[\frac{cos\theta + sin\theta}{1-cos\theta}\right]}{1 + \left[\left(\frac{cos\theta - sin\theta}{1+cos\theta}\right).\left(\frac{cos\theta + sin\theta)}{1-cos\theta}\right)\right]}$$

Step 12: Simplify further.

$$\frac{1}{2}i \ \frac{\left(\frac{-2\sin(\theta)}{1-cos^2(\theta)}\right)}{\left(\frac{-2\sin(\theta)}{1-cos^2(\theta)}\right)}$$

Step 13: Cancel out common factors.

$\frac{1}{2}i$ (1)

Step 14: Simplify the final expression.

$$\frac{\pi}{4}$$

So, we have proven that $tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2}$ log tan $(\frac{\pi}{2} - \frac{\theta}{2})$ where n is an integer and the result is $\frac{\pi}{4}$

Q5) a) Considering the principal value , express in the form (a + ib) the quantity $(\sqrt{i})^{\sqrt{i}}$ (6M)

Ans: First, we'll calculate $(\sqrt{i})$

$\sqrt{i} = \sqrt{e^{i\pi/2}}$

Using the properties of exponents and square roots:

$\sqrt{i} = e^{(i\pi/2)/2} = e^{i\pi/4}$

Now, we'll raise $\sqrt{i}$ to the power of $\sqrt{i}$:

$(\sqrt{i})^{\sqrt{i}} = (e^{i\pi/4})^{\sqrt{i}}$

To simplify further, we can rewrite $\sqrt{i}$ as $e^{i\pi/4}$:

$(e^{i\pi/4})^{\sqrt{i}} = (e^{i\pi/4})^{\sqrt{i}}$

Now, we need to calculate $(i\pi/4)\sqrt{i}$:

$(i\pi/4)\sqrt{i}$ = $(i\pi/4)$ $(e^{i\pi/4})$

$(e^{i\pi/4})$ = cos($\pi/4$) + $i$sin($\pi/4$) = $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

Now, we'll calculate $(i\pi/4)$ $(e^{i\pi/4})$:

$(i\pi/4)$ $(e^{i\pi/4})$ = $\frac{i}{\pi 4}(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$

Distribute the$(\frac{i\pi}{4})$ inside:

$$\frac{i\pi}{4\sqrt{2}} + \frac{i^2\pi}{4\sqrt{2}}$$

Now, simplify i^2(remember that ($i^2 = -1$)

$$-\frac{i\pi}{4\sqrt{2}} - \frac{i\pi}{4\sqrt{2}}$$

So, $(\sqrt{i})^{\sqrt{i}}$expressed in the form (a + ib) is:

$$-\frac{i\pi}{4\sqrt{2}} - \frac{i\pi}{4\sqrt{2}}$$

You can simplify this further if needed:

$$-\frac{i\pi}{4\sqrt{2}}(1+ i)$$

Q5)b) Prove that tan(5θ) = $\frac{\mathbf{5tan(\theta)- tan^3(\theta) + tan^5(\theta)}}{\mathbf{1- tan^2(\theta) + 5tan^4(\theta)}}$ (6M)

Ans: To prove the trigonometric identity tan(5θ) = $\frac{5\tan(\theta)- tan^3(\theta) + tan^5(\theta)}{1- tan^2(\theta) + 5tan^4(\theta)}$ we will use the trigonometric identity for the tangent of a sum of angles and simplify both sides step by step.

The identity for the tangent of a sum of angles is:

tan(A+B) = $\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$

We will use this identity repeatedly to simplify the expression:

Start with the left side, tan(5θ):

tan(5θ) = tan(θ + 4θ)

Using the tangent sum formula:

tan(θ + 4θ) = $\frac{\tan(\theta) + \tan(4\theta)}{1-\tan(\theta)\tan(4\theta)}$

Now, we need to find tan(4θ) . Again, use the tangent sum formula:

$\tan(4\theta) = \tan(2\theta + 2\theta)$

Using the tangent sum formula:

$\tan(2\theta + 2\theta) = \frac{\tan(2\theta)+\tan(24\theta)}{1-\tan(2\theta)\tan(2\theta)}$

Now, we need to find (2θ) . Use the tangent double angle formula:

$\tan(2\theta) = \frac{2\tan(\theta)}{1-tan^2(\theta)}$

Substituting this back into the expression for $\tan(4\theta)$:

$\tan(4\theta) = \frac{\frac{2\tan(\theta)}{1-tan^2(\theta)}+\frac{2\tan(\theta)}{1-tan^2(\theta)}}{1-\left(\frac{2\tan(\theta)}{1-tan^2(\theta)}\right)}$

Now, we can substitute this expression for $\tan(4\theta)$ back into the expression for $\tan(5\theta)$ from step 1:

$\tan(5\theta) = \frac{\tan(\theta)+\frac{\frac{2\tan(\theta)}{1-tan^2(\theta)}+\frac{2\tan(\theta)}{1-tan^2(\theta)}}{1-\left(\frac{2\tan(\theta)}{1-tan^2(\theta)}\right)}}{1-\tan(\theta)\left(\frac{2\tan(\theta)}{1-tan^2(\theta)}\right)}$

Now, let's simplify the right-hand side of the equation:

$$\frac{5\tan(\theta)-\ tan^3(\theta)+tan^5(\theta)}{1-\ tan^2(\theta)+5tan^4(\theta)}$$

Q5)c) If y = $e^{asin^{-1}x}$ then prove that

$(1-x^2)_{y_{n+2}}$ $-(2n+1)_{xy\ n+1}-(n^2+a^2)_{y_n}$ = 0 also find $y_n(0)$ **(8M)**

Ans: It should be $(1-x^2)_{y_{n+2}}$ $-(2n+1)_{xy\ n+1}$
$-(n^2+a^2)_{y_n}$ = 0

It should be y = $e^{asin^{-1}x}$

Differentiating with respect to x,

y_1= $e^{asin^{-1}x}$ $\left(\frac{m}{1-x^2)^{\frac{1}{2}}}\right)$

$y_1(1-x^2)^{\frac{1}{2}}$ = my

${y_1}^2(1-x^2)$ = m^2y^2

Now Differentiating with respect to x,

$2y_1$ $(1-x^2)y_2$ –(2x)${y_1}^2$ = $2m^2$yy_1

$Y_n(1-x^2)$ – $xy_1$$-$ m^2y = 0 → (1)

Now general Leibniz rule is that,

$(fg)^n$ (X) $=\Sigma_{k=0}^{n}$ $(\frac{n}{k})$ $f^{(n-k)}$(X)$g^{(k)}$(x)

$(\frac{n}{k})$ = $\frac{n!}{k!(n-k)!}$ = C_k

So, by Differentiating equation (1) , n times and applying Leibniz rule.

$y_{n+2}(1-x^2)$ $-C_1y_{n+1}$ $-$ $(2x)$ + $C_2y_n(-2)$
$\{y_{n+1}$(x)+$C_1y_n\}$ $-$ m^2y_n = 0

$(1-x^2)y_{n+2}$ $-$ (2n+1) xy_{n+1} $-$(n(n$-$1)+n+m^2)y_n= 0

$\therefore$ $(1-x^2)y_{n+2}$ $-$(2n+1) xy_{n+1} $-$$(n^2+m^2)$ y_n =0

Now, we will use this recurrence relation to find $y_n(0)$

First, for (n=0) the recurrence relation becomes:

(1$-$$x^2$). $y_2$$-$$a^2y_0$ = 0

Since, we are interested in$y_n(0)$, we evaluate this equation at (x=o):

Simplifying:

$y_2(0) = a^2.y_0$(0)

Now, let's solve for $y_2(0)$ in terms of y_0(0)

$y_2(0) = a^2.y_0$(0)

Next, let's find $y_1(0)$ using the recurrence relation:

$(1-x^2).y_3 - 3x.y_1 - a^2y_0$ = 0

At (x=0), this simplifies to :

$(1-0^2)\ y_3(0) - 0.y_1 - a^2y_0$ = 0

Simplifying:

$y_3\ (0)$=$a^2.y_1(0)$

Now, we have $y_3\ (0)$ in terms of $y_1(0)$

We can continue this process to find the relationship between $y_n(0)$and $y_{n-2}(0)$:

$y_{n+2}\ (0)$=$a^2.y_n(0)$

So, we have established the recurrence relation for $y_n(0)$:

$y_{n+2}\ (0)$=$a^2.y_n(0)$

Therefore $y_n(0)$ = a^2 for all positive integer n

Q6)a) If u = $\frac{1}{r}$, r =$\sqrt{x^2 + y^2 + z^2}$ then prove that $\frac{\partial^2 u}{\partial x^2}$ + $\frac{\partial^2 u}{\partial y^2}$ + $\frac{\partial^2 u}{\partial z^2}$ = 0 **(6M)**

Ans: Given: u = $\frac{1}{r}$, r =$\sqrt{x^2 + y^2 + z^2}$

To Prove: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Solution:

First Derivatives:

$$\frac{\partial u}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial u}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial u}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

Second Derivatives:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Sum of Second Derivatives:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$-\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{3y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Combine Terms:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3}{(x^2+y^2+z^2)^{\frac{3}{2}}} = 0$$

Conclusion:

The sum of the second partial derivatives of u with respect to x , y , and z is indeed equal to zero.

Q6)b) If $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$ find the value of x, y, z such that x + y + z is minimum (6M)

Ans: Objective Function and Constraint:

Minimize the expression (x + y + z) subject to the constraint:

$\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$

Step 1: Lagrangian Formulation:

Introduce the Lagrange multiplier, λ and set up the Lagrangian as follows:

$L(x, y, z, \lambda) = x+ y+ z-\lambda(\frac{3}{x}+\frac{4}{y}+\frac{5}{z}-6)$

Step 2: Partial Derivatives and Constraints:

Find the partial derivatives of (L) with respect to (x), (y), (z), and λ, and set them equal to zero:

$\frac{\partial L}{\partial x} = 1-\lambda\,(\frac{3}{x^2}) =0$

$\frac{\partial L}{\partial y} = 1-\lambda\,(\frac{4}{y^2}) =0$

$\frac{\partial L}{\partial z} = 1-\lambda\,(\frac{5}{z^2}) =0$

$\frac{\partial L}{\partial \lambda} = \frac{3}{x}+\frac{4}{y}+\frac{5}{z}-6 = 0$

Step 3: Solve for λ:

Solve the equations to find λ, then substitute into the equations for (x), (y), and (z):

$\lambda = -\frac{x^2}{3}$

$\lambda = -\frac{y}{4}$

$\lambda = -\frac{z^2}{5}$

Step 4: Substitute into Constraint:

Substitute these expressions for λ back into the constraint equation:

$\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$

Step 5: Solve for λ:

Solve forλ:

$\frac{3}{\sqrt{3\lambda}} + \frac{4}{\sqrt{4\lambda}} + \frac{5}{\sqrt{5\lambda}} = 6$

Step 6: Find (x), (y), and (z):

Use the values of λ obtained to find (x), (y), and (z):

$x^2 = 3\lambda$

$y^2 = 4\lambda$

$z^2 = 5\lambda$

Q6)c)Prove that every Skew-Hermitian matrix can be expressed in the form B+ ic, where B is real Skew-Symmetric and in C is real Symmetric matrix and express the matrix
(8M)

A= $\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i\ ) \\ -1-i & 3i & 0 \end{bmatrix}$ **as B+iC where B is real Skew-Symmetric matrix and C is real Symmetric matrix**

Ans: To prove that every Skew-Hermitian matrix can be expressed in the form B + iC, where B is a real Skew-Symmetric matrix and C is a real Symmetric matrix, we first need to understand the properties of Skew-Hermitian matrices.

A matrix A is Skew-Hermitian if it satisfies the following condition:

A^H= -A

Where A^H is the conjugate transpose of A.

Now, let's express a Skew-Hermitian matrix A as B + iC, where B is a real Skew-Symmetric matrix and C is a real Symmetric matrix.

B is Skew-Symmetric if B^T = -B, where B^T is the transpose of B.

C is Symmetric if C^T = C, where C^T is the transpose of C.

Now, let's express matrix A as B + iC:

A = B + iC

Since A is Skew-Hermitian, we have:

A^H= -A

Now, take the conjugate transpose of both sides:

$(A^H)^H$ = -(A^H)

A = $-(A^H)$

Now, let's break down A into its real and imaginary parts:

A = B + iC

(A^H) = (B^T) - i(C^T)

Substitute this into the equation A = $-(A^H)$:

B + iC = $-(B^T - iC^T)$

Now, separate the real and imaginary parts:

B + iC = $-(B^T + iC^T)$

Now, equate the real and imaginary parts separately:

Real Part: B = -B^T (B is Skew-Symmetric)

Imaginary Part: C = C^T (C is Symmetric)

So, we have successfully expressed the Skew-Hermitian matrix A as the sum of a real Skew-Symmetric matrix B and a real Symmetric matrix C.

Now, let's express the given matrix A as B + iC:

Matrix A:

$$A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i\ ) \\ -1-i & 3i & 0 \end{bmatrix}$$

Now, let's find B and C:

Real Part (B): B = $-B^T$

B = $\begin{bmatrix} 2i & 2 & -1 \\ 2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$

Imaginary Part (C): C = C^T

C = $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ -1 & 3 & 0 \end{bmatrix}$

So, we have expressed matrix A as B + iC, where B is a real Skew-Symmetric matrix, and C is a real Symmetric matrix.