MUMBAI UNIVERSITY

SEMESTER – 1
APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1 (a) Show that $\sec h^{-1}(\sin \theta) = \log \cot\left(\frac{\theta}{2}\right)$. [3]

ANS: LHS = $\sec h^{-1}(\sin \theta)$

Let $y = \sec h^{-1}(\sin \theta)$

$\sec hy = \sin \theta$

$\frac{1}{\sin\theta} = \frac{1}{\sec hy}$

$\cos hy = \operatorname{cosec} \theta$

$y = \cos h^{-1}(\operatorname{cosec} \theta)$

but $\cos h^{-1}x = \log\left|x + \sqrt{x^2-1}\right|$

$\therefore y = \log\left|\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right|$

$\therefore y = \log\left|\operatorname{cosec}\theta + \cot\theta\right|$

$= \log\left|\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right|$

$= \log\left|\frac{1+\cos\theta}{\sin\theta}\right|$

$= \log\left|\frac{2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\right|$

$= \log\left|\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right|$

$= \log\cot\left(\frac{\theta}{2}\right)$.

= RHS

$\therefore$ LHS = RHS

Hence proved

(b)Show that a matrix A = $\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$ is unitary. [3]

ANS: Given $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

To prove unitary, we have to prove $AA^{\theta} = I$

$\therefore A^{\theta} = \frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

$\therefore$ LHS = AA^{θ}

$= \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 2+2+0 & -2i+2i+0 & 0+0+0 \\ 2i-2i+0 & 2+2+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

LHS= I

= RHS

LHS =RHS

Hence proved.

(c)Evaluate $\lim_{x\to 0} \sin x \log x$. [3]

ANS: Let L = $\lim_{x\to 0} \sin x \log x$

L = $\lim_{x\to 0} \frac{\log x}{cosec\, x}$

L = $\lim_{x\to 0} \frac{\frac{1}{x}}{-cosec\, x \cot x}$ ……………………… (By L hospital)

L = $\lim_{x\to 0} \frac{-\sin x \tan x}{x}$

L = $\lim_{x\to 0} -\tan x$

L = 0

(d)Find the n^{th} derivative of y=$e^{ax}\cos^2 x \sin x$. [3]

ANS: Given y=$e^{ax}\cos^2 x \sin x$

y= $e^{ax}\left(\frac{1+\cos 2x}{2}\right)$ sin x

y= $\frac{1}{2}(e^{ax}\sin x + e^{ax}\cos 2x \sin x)$

y= $\frac{1}{2}\left(e^{ax}\sin x + \frac{1}{2}e^{ax}(\sin 3x - \sin x)\right)$

y= $\frac{1}{2}\left(\frac{1}{2}e^{ax}\sin 3x + \frac{1}{2}e^{ax}\sin x\right)$

Diff n times,

$y_n = \frac{1}{2}\left(\frac{1}{2}e^{ax}(\sqrt{a^2+9})^n \sin(3x + n\tan^{-1}\frac{3}{a}) + \frac{1}{2}e^{ax}(\sqrt{a^2+1})^n \sin(x + n\tan^{-1}\frac{1}{a})\right)$.

(e)If x = r cos θ and y= r sin θ prove that JJ^{-1}=1. [4]

ANS: Given x =r cos θ and y=r sin θ i.e. x,y → f (r,Θ)

$\frac{\partial x}{\partial r} = \cos\Theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$\therefore J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\Theta + \sin^2\Theta) = r.$

$\therefore J = r$…………………… (1)

Now, to find values of r and Θ

$\therefore x^2 + y^2 = r^2(\cos^2\Theta + \sin^2\Theta) = r^2$

$\therefore r = \sqrt{x^2 + y^2}$ and $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\Theta$

$\therefore \Theta = \tan^{-1}\frac{y}{x}$

$\therefore J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$

$= \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$

$= \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}}$

$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$ ………………. (2)

From 1 and 2, we get

Hence, $JJ' = r.\frac{1}{r} = 1$

Hence proved

(f)Using coding matrix A=$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ encode the message THE CROW FLIES AT MIDNIGHT. [4]

ANS:

T = 20 H = 8 E = 5 C = 3 R = 18 O = 15 W = 23 F = 6 L = 12 I = 9 E = 5
S = 19 A = 1 T = 20 M = 13 I = 9 D= 4 N = 14 I = 9 G = 7 H = 8 T=20

C = AB

$B = \begin{bmatrix} 20 & 5 & 18 & 23 & 12 & 5 & 1 & 13 & 4 & 9 & 8 \\ 8 & 3 & 15 & 6 & 9 & 19 & 20 & 9 & 14 & 7 & 20 \end{bmatrix}$

$\therefore \quad C = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 20 & 5 & 18 & 23 & 12 & 5 & 1 & 13 & 4 & 9 & 8 \\ 8 & 3 & 15 & 6 & 9 & 19 & 20 & 9 & 14 & 7 & 20 \end{bmatrix}$

$C = \begin{bmatrix} 48 & 13 & 51 & 52 & 33 & 29 & 22 & 35 & 22 & 25 & 36 \\ 68 & 18 & 69 & 75 & 45 & 34 & 23 & 48 & 26 & 34 & 44 \end{bmatrix}$

muquestionpapers.com

Q.2] (a) Find all values of $(1+i)^{\frac{1}{3}}$ and show that their continued product is (1 + i). [6]

ANS: Let Z = $(1+i)^{\frac{1}{3}}$

$Z = [\sqrt{2}\ (cos\ \frac{\pi}{4} + isin\frac{\pi}{4})]^{\frac{1}{3}}$

$Z = (\sqrt{2})^{\frac{1}{3}} . [\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin(2k\pi + \frac{\pi}{4})]^{\frac{1}{3}}$

$Z = 2^{\frac{1}{6}}[\cos\ (\frac{8k\pi+\pi}{12}) + i\ \sin(\frac{8k\pi+\pi}{12})]$

Putting k = 0, 1, 2.

$Z_0 = 2^{\frac{1}{6}} . e^{\frac{i\pi}{12}}$

$Z_1 = 2^{\frac{1}{6}} . e^{\frac{9i\pi}{12}}$

$Z_2 = 2^{\frac{1}{6}} . e^{\frac{17i\pi}{12}}$

$\therefore Z_0\ Z_1\ Z_2 = 2^{\frac{3}{6}} . e^{\frac{27i\pi}{12}}$

$= 2^{\frac{1}{2}} . e^{\frac{9i\pi}{4}}$

$= \sqrt{2}\ (\ cos\ \frac{9\pi}{4} + isin\frac{9\pi}{4}\ )$

$= \sqrt{2}\ (\ \frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}})$

$= (1 + i\ ).$

(b)Find the non-singular matrices P & Q such that PAQ is in normal form where $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$. [6]

ANS. Given matrix is A= $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

The order of matrix is 3 x 4

$\therefore A = I_3\ A\ I_4.$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$

$$\begin{bmatrix}1 & 2 & 3 & 4\\0 & -3 & -2 & -5\\0 & -6 & -4 & -22\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-3 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Operate C_2-$2C_1$; C_3-$3C_1$; C_4-$4C_1$

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & -3 & -2 & -5\\0 & -6 & -4 & -22\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-3 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & -2 & -3 & -4\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Operate $\frac{C_2}{-3};\frac{C_3}{-2}$

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 1 & -5\\0 & 2 & 2 & -22\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-3 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & \frac{2}{3} & \frac{3}{2} & -4\\0 & \frac{-1}{3} & 0 & 0\\0 & 0 & \frac{-1}{2} & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Operate R_3-$2R_2$

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & -5\\0 & 0 & 0 & -12\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\1 & -2 & 1\end{bmatrix}A\begin{bmatrix}1 & \frac{2}{3} & \frac{3}{2} & -4\\0 & \frac{-1}{3} & 0 & 0\\0 & 0 & \frac{-1}{2} & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Operate C_{34}

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 1 & 0\\0 & 0 & -12 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\1 & -2 & 1\end{bmatrix}A\begin{bmatrix}1 & \frac{2}{3} & \frac{-2}{3} & \frac{5}{6}\\0 & \frac{-1}{3} & \frac{-5}{3} & \frac{1}{3}\\0 & 0 & 0 & \frac{-1}{2}\\0 & 0 & 1 & 0\end{bmatrix}$$

Operate $\frac{R_3}{-12}$

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\\frac{-1}{12} & \frac{1}{6} & \frac{-1}{12}\end{bmatrix}A\begin{bmatrix}1 & \frac{2}{3} & \frac{-2}{3} & \frac{5}{6}\\0 & \frac{-1}{3} & \frac{-5}{3} & \frac{1}{3}\\0 & 0 & 0 & \frac{-1}{2}\\0 & 0 & 1 & 0\end{bmatrix}$$

$[I_3, 0]$ = PAQ ie PAQ is in normal form, Where, $P=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-1}{12} & \frac{1}{6} & \frac{-1}{12} \end{bmatrix}$ and $Q=\begin{bmatrix} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{5}{6} \\ 0 & \frac{-1}{3} & \frac{-5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(c)Find maximum and minimum values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. [8]

ANS: Given $f(x) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$… (1)

STEP 1] for maxima, minima, $\frac{\partial f}{\partial x} = 0; \frac{\partial f}{\partial y} = 0$

$3x^2+3y^2-30x+72=0$ and $6xy - 30y=0$

$\therefore$ y (6x-30) =0

y=0, x= 5

For x=5; From Equation $3x^2+3y^2-30x+72=0$, we get $y^2-1=0$

$Y=\pm 1$

Hence (4,0) , (6,0) , (5,1) ,(5,-1) are the stationary points.

STEP 2] Now, $r=\frac{\partial^2 f}{\partial x^2}=6x-30$;

$S=\frac{d^2 f}{\partial x \partial y}=6y$;

$t=\frac{\partial^2 f}{\partial y^2}=6x-30$

STEP 3] for $(x, y) \equiv (4, 0)$, r=-6, s=0, t= -6;

$rt - s^2=(-6)(-6)-0=36>0$ and $r< 0$.

This shows that the function is maximum at (4, 0)

$\therefore$From Equation (1)

$F_{max}=f(4, 0) =4^3+0-15(4^2)+0+72(4) =64 - 240 + 288$

$F_{max}=112$

STEP 4] For $(x,y)\equiv(6,0)$

r=6, s=0, t=6

$rt-s^2=36$ but r=6>0

This shows that function is minimum at (6, 0).

From Equation (1),

$F_{min}=f(6,0)=6^3+0-15(6)^2+0+72(6)=108$.

STEP 5] For$(x, y) \equiv (5, 1)$

r=0, s=6, t=0;

$(rt-s^2)<0$

This shows that at (5, 1) and (5,-1) function is **neither maxima nor minima**.

These points are **saddle points.**

muquestionpapers.com

Q.3] (a) If u=e^{xyz} f $(\frac{xy}{z})$ prove that $x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}$=2xyzu and y$\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}$=2xyzu and hence show that $x\frac{\partial^2 u}{\partial z\partial x}=y\frac{\partial^2 u}{\partial z\partial y}$. [6]

ANS: U=e^{xyz} f $(\frac{xy}{z})$

$$\frac{\partial u}{\partial x}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\times\frac{y}{z}\right]+f\left(\frac{xy}{z}\right)[e^{xyz}\times xy]$$

$$x\frac{\partial u}{\partial x}=e^{xyz}\left[\frac{xy}{z}f'\left(\frac{xy}{z}\right)+xyzf(\frac{xy}{z})\right]\ \ldots\ldots\ldots\ldots\ldots\ (1)$$

$$\frac{\partial u}{\partial y}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\times\frac{x}{z}\right]+f\left(\frac{xy}{z}\right)[e^{xyz}\times xz]$$

$$y\frac{\partial u}{\partial y}=e^{xyz}\left[\frac{xy}{z}f'\left(\frac{xy}{z}\right)+xyzf(\frac{xy}{z})\right]\ \ldots\ldots\ldots\ldots\ldots\ (2)$$

$$\frac{\partial u}{\partial z}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\times\frac{xy}{z^2}\right]+f\left(\frac{xy}{z}\right)[e^{xyz}\times xy]$$

$$z\frac{\partial u}{\partial z}=e^{xyz}\left[-\frac{xy}{z}f'\left(\frac{xy}{z}\right)+xyzf(\frac{xy}{z})\right]\ \ldots\ldots\ldots\ldots\ldots\ (3)$$

Adding 1 and 3, we get

$$x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=2e^{xyz}\,xyzf\left(\frac{xy}{z}\right)=2xyzu$$

Adding 2 and 3, we get

$$y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2e^{xyz}\,xyzf\left(\frac{xy}{z}\right)=2xyzu$$

For deduction,

$$x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=2xyzu$$

Diff w.r.t z

$$x\frac{\partial^2 u}{\partial x\partial z}+\left[z\frac{\partial^2 u}{\partial z}+\frac{\partial u}{\partial z}(1)\right]=2xy[z\frac{\partial u}{\partial z}+u(1)]$$

$$x\frac{\partial^2 u}{\partial x\partial z}=(2xyz-1)\frac{\partial u}{\partial z}-z\frac{\partial^2 u}{\partial z^2}+2xyu\ \ldots\ldots\ldots\ldots\ldots\ (4)$$

Similarly,

$$y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2xyzu$$

Diff w.r.t z and solving, we get

$$y\frac{\partial^2 u}{\partial y\partial z}=(2xyz-1)\frac{\partial u}{\partial z}-z\frac{\partial^2 u}{\partial z^2}+2xyu\ \ldots\ldots\ldots\ldots\ldots\ (5)$$

∴ From 4 and 5, we get

$$x\frac{\partial^2 u}{\partial z\partial x}=y\frac{\partial^2 u}{\partial z\partial y}.$$

(b)By using Regular falsi method solve 2x – 3sin x – 5 = 0. [6]

ANS: Let f(x) = 2x – 3sin x - 5

f(1) = -5

f(2) = -5.5244

f(3) = -3.7379 < 0

f(4) = 0.5766 > 0

∴Roots lies between 2 and 3

muquestionpapers.com

Iteration	a	b	f(a)	f(b)	x = $\frac{af(b)-bf(a)}{f(b)-f(a)}$	f(x)
I.	2	3	-3.7279	0.5766	2.8660	-0.0841
II.	2.866	3	-0.0841	0.5766	2.8831	-0.0009
III.	2.8831	3	0.0009	0.5766	2.8833	-

(c)if y=sin[log(x^2+2x+1)] then prove that $(x+1)^2y_{n+2}$ +(2n +1)(x+ 1)y_{n+1} + (n^2+4)y_n=0. **[8]**

ANS: We have,

$y=\sin[\log(x^2+2x+1)]$

Diff with x

$y_1 = \cos[\log(x^2+2x+1)] \times \frac{1}{x^2+2x+1} \times (2x+2)$

$y_1 = \cos[\log(x^2+2x+1)] \times \frac{2(x+1)}{(x+1)^2}$

$y_1 = \cos[\log(x^2+2x+1)] \times \frac{2}{x+1}$

$(x+1)\,y_1 = 2\cos[\log(x^2+2x+1)]$

Diff with x again,

$(x+1)\,y_2 + y_1 = -2\sin[\log(x^2+2x+1)] \times \frac{1}{x^2+2x+1} \times (2x+2)$

$(x+1)\,y_2 + y_1 = -2\sin[\log(x^2+2x+1)] \times \frac{2(x+1)}{(x+1)^2}$

$(x+1)^2\,y_2 + (x+1)y_1 = -4\sin[\log(x^2+2x+1)]$

$(x+1)^2\,y_2 + (x+1)y_1 = -4y$

By Leibnitz Theorem,

$\left[y_{n+2}(x+1)^2 + ny_{n+1}.2(x+1) + \frac{n(n-1)}{2!}y_n.2\right] + [y_{n+1}.(x+1) + ny_n(1)$] = $-4y_n$

$y_{n+2}(x+1)^2 + (2n+1)(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0$

Q.4] (a) State and prove Euler's Theorem for three variables. **[6]**

ANS:

Statement: If u=f(x, y, z) is a homogeneous function of degree n, then -

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

Let, u=f(x, y, z) is a homogeneous function of degree n.

Putting X = x t, Y = y t, Z = z t.

f(X,Y,Z) = t^n f(x,y,z) ………. (1)

Diff LHS w.r.t t,

muquestionpapers.com

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial X}.\frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial t} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial t}$$

$$\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial X}. + y\frac{\partial f}{\partial Y} + z\frac{\partial f}{\partial Z} \ldots\ldots \text{(2)}$$

Diff RHS w.r.t. t,

$$\frac{\partial f}{\partial t} = nt^{n-1}f(x, y, z)$$

Now put t = 1, we get $\frac{\partial f}{\partial t} = nf(x, y, z)$ ……… (3)

From equation 2 and 3, we get

$$x\frac{\partial f}{\partial X}. + y\frac{\partial f}{\partial Y} + z\frac{\partial f}{\partial Z} = nf(x, y, z)$$

$$x\frac{\partial f}{\partial X}. + y\frac{\partial f}{\partial Y} + z\frac{\partial f}{\partial Z} = nu$$

Hence proved

(b) By using De Moirés Theorem obtain tan 5θ in terms of tan θ and show that 1 - 10 $\tan^2(\frac{\pi}{10})$ + 5$\tan^4(\frac{\pi}{10})$ = 0. [6]

ANS: (cos 5θ + isin 5θ) = (cos θ + isin θ) ^{5}

= cos ^{5}θ + 5cos ^{4}θ isin θ + 10 cos ^{3}θ i^2 sin ^{2}θ + 10 cos ^{2}θ i^3 sin ^{3}θ + 5 cos θ i^4 sin ^{4}θ + i^5 sin ^{5}θ

= cos ^{5}θ + 5cos ^{4}θ isin θ - 10 cos ^{3}θ sin ^{2}θ – 10i cos ^{2}θ sin ^{3}θ + 5 cos θ sin ^{4}θ + i sin ^{5}θ

= (cos ^{5}θ- 10 cos ^{3}θ sin ^{2}θ + 5 cos θ sin ^{4}θ) + i (5cos ^{4}θ sin θ – 10 cos ^{2}θ sin ^{3}θ + sin ^{5}θ)

Equating both sides we get,

∴ cos 5θ = cos ^{5}θ- 10 cos ^{3}θ sin ^{2}θ + 5 cos θ sin ^{4}θ

∴ sin 5θ = 5cos ^{4}θ sin θ – 10 cos ^{2}θ sin ^{3}θ + sin ^{5}θ

But tan 5θ = sin 5θ / cos 5θ

= (5cos ^{4}θ sin θ – 10 cos ^{2}θ sin ^{3}θ + sin θ) / (cos ^{5}θ- 10 cos ^{3}θ sin ^{2}θ + 5 cos θ sin ^{4}θ)

Dividing by cos ^{5}θ

tan 5θ $= \frac{5\tan\theta \; -10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

for deduction, put θ = $\frac{\pi}{10}$

cot 5x$\frac{\pi}{10}$ $= \frac{1 - 10\tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10}}{5\tan\frac{\pi}{10} \; -10\tan^3\frac{\pi}{10} + \tan^5\frac{\pi}{10}}$

∴ 1 - 10 $\tan^2(\frac{\pi}{10})$ + 5$\tan^4(\frac{\pi}{10})$ = 0.

Hence proved.

(c)Investigate for what values of λ and μ the equations [8]

2x + 3y + 5z = 9

7x + 3y - 2z = 8

$2x + 3y + \lambda z = \mu$ have

A. No solutions

B. Unique solutions

C. An infinite number of solutions.

ANS: Consider the system of equation of

$2x + 3y + 5z = 9$

$7x + 3y - 2z = 8$

$2x + 3y + \lambda z = \mu$

The above system is given as Ax=B

$$\begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ \mu \end{bmatrix}$$

Where $A = \begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 2 & 3 & \lambda \end{bmatrix}$ $\qquad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

And $B = \begin{bmatrix} 8 \\ 9 \\ \mu \end{bmatrix}$

$R_3 - R_2$

$$\begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ \mu - 9 \end{bmatrix}$$

(A) For no solution,

$\rho(A) \neq \rho(A\ B)$

$\therefore \lambda - 5 = 0$ and $\mu - 9 \neq 0$

$\therefore \lambda = 5$ and $\mu \neq 9$

(B) For a unique solution

$\rho(A) = \rho(A\ B) = 3$

$\therefore \lambda - 5 \neq 0$ and μ may be anything

$\therefore \lambda \neq 5$ for all values of μ

(C) For infinite solutions,

$\rho(A) = \rho(A\ B) < 3$

$\therefore \lambda - 5 = 0$ and $\mu - 9 = 0$

$\therefore \lambda = 5$ and $\mu = 9$

Q.5] (a) Find n^{th} derivative of $\frac{1}{x^2+a^2}$. **[6]**

ANS: $y = \frac{1}{x^2+a^2}$.

$y = \frac{1}{(x+ai)(x-ai)}$.

Let $\frac{1}{(x+ai)(x-ai)} = \frac{A}{(x+ai)} + \frac{B}{(x-ai)}$

$1 = A(x - ai) + B(x + ai)$

muquestionpapers.com

Put x = ai,

1 = B(2ai)

B = $\frac{1}{2ai}$

Put x = -ai, we get

A = $-\frac{1}{2ai}$

$\therefore$ y = $\frac{\frac{1}{-2ai}}{(x+ai)} + \frac{\frac{1}{2ai}}{(x-ai)}$

$\therefore$ y = $\frac{1}{2ai}\left[\frac{1}{(x+ai)} - \frac{1}{(x-ai)}\right]$

Diff n times, we get

$y_n = \frac{1}{2ai}\left[\frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}}\right].$

(b) If z = f (x, y) where $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ then prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$. [6]

ANS: Given: z = f (x, y) , $x = e^u + e^{-v}$ ……… (1)

$y = e^{-u} - e^v$ ………… (2)

By Chain Rule,

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ ……… (3)

And

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ ……….. (4)

$\therefore$ From equation 1 and 2,

$\frac{\partial x}{\partial u} = e^u$ $\frac{\partial x}{\partial v} = -e^{-v}$

$\frac{\partial y}{\partial u} = -e^{-u}$ $\frac{\partial y}{\partial v} = -e^v$

$\therefore$ From equation 3 and 4,

$\frac{\partial z}{\partial u} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$ ………. (5)

And

$\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$ ………… (6)

By Subtracting Equation 5 and 6,

$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$

$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ …………………….. (By using equation 1 and 2)

$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Hence proved.

(c) Solve using Gauss Jacobi Iteration method [8]

$2x + 12y + z - 4w = 13$

muquestionpapers.com

$13x$ + 5y - 3z + w = 18
$2x$ + y – 3z + 9w = 31
$3x$ - 4y + 10z + w = 29

ANS:

$x = \frac{18-5y+3z-w}{13}$

$y = \frac{13-2x-z+4w}{12}$

$z = \frac{29-3x+4y-w}{10}$

$w = \frac{31-2x-y+3z}{9}$

Iteration	x	y	z	w
1	1.3846	1.0833	2.9	3.4444
2	1.3722	1.7590	2.5735	3.9831
3	0.9956	1.9679	2.7936	3.8019
4	0.9800	1.9519	3.0083	3.9357
5	1.0254	1.9812	2.9932	4.0126
6	1.0047	2.0005	2.9836	3.9942
7	0.9965	1.9987	2.9994	3.9934
8	1.0009	1.9984	3.0012	4.0007
9	1.0008	2.0000	2.9990	4.0004
10	0.9997	2.0001	2.9997	3.9995
11	0.9999	1.9999	3.0002	4.0000
12	1.0001	2	3	4.0001
13	1	2	3	4

∴ x = 1, y = 2, z = 3, w = 4.

Q.6] (a) If y = log [tan $\left(\frac{\pi}{4}+\frac{x}{2}\right)$] Prove that **[6]**

I. **tan h $\frac{y}{2} = \tan\frac{x}{2}$**

II. **cos hy cos x = 1**

ANS: 1] sin h$\frac{y}{2}$ = $\frac{e^{\frac{y}{2}}-e^{-\frac{y}{2}}}{2}$

cos h$\frac{y}{2}$ = $\frac{e^{\frac{y}{2}}+e^{\frac{-y}{2}}}{2}$

tan h $\frac{y}{2}$ = $\frac{\sin h\frac{y}{2}}{\cos h\frac{y}{2}}$

muquestionpapers.com

$$= \frac{\frac{e^{\frac{y}{2}} - e^{\frac{-y}{2}}}{2}}{\frac{e^{\frac{y}{2}} + e^{-\frac{y}{2}}}{2}}$$

$$= \frac{e^{y} - 1}{e^{y} + 1}$$

But $e^{u} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$

$$\therefore \tan h\, \frac{y}{2} = \frac{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} - 1}{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + 1}$$

$$= \frac{1+\tan\frac{x}{2} - 1 + \tan\frac{x}{2}}{1+\tan\frac{x}{2} + 1 - \tan\frac{x}{2}}$$

$$= \tan\frac{x}{2}$$

$\therefore \tan h\, \frac{y}{2} = \tan\frac{x}{2}$

2] $y = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

$e^{y} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}$$

$$e^{y} = \frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}$$

$$e^{-y} = \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

$\cos hy = \frac{e^{y} + e^{-y}}{2}$

$$= \frac{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + \frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}{2}$$

$$= \frac{(1+\tan\frac{x}{2})^2 + (1-\tan\frac{x}{2})^2}{2(1-\tan^2\frac{x}{2})}$$

$$= \frac{1+2\tan\frac{x}{2} + \tan^2\frac{x}{2} + 1 + \tan^2\frac{x}{2} - 2\tan\frac{x}{2}}{2(1-\tan^2\frac{x}{2})}$$

$$= \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

$\therefore$ cos hy = $\frac{1}{\cos x}$

$\therefore$ cos hy cos x = $\frac{1}{\cos x}$ .cos x

cos hy cos x = 1

Hence proved

(b) If u = $\sin^{-1}\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$ prove that

$x^2\frac{\partial^2 u}{\partial^2 x}$ + 2xy $\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y} = \frac{\tan u}{144}$ [tan ²u + 13]. [6]

ANS:

Given u = $\sin^{-1}\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$

z = sin u = $\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$ is homogeneous function in x and y with degree -$\frac{1}{12}$

$\therefore$We have the result,

If z = f (u) is homogeneous function of degree x and y then

$x^2\frac{\partial^2 u}{\partial^2 x}$ + $2xy\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y}$ = g (u) [g ′(u) – 1] where g(u) = n $\frac{f(u)}{f'(u)}$.

n = -$\frac{1}{12}$, f (u) = sin u, f ′(u) = cos u

$\therefore$g (u) = -$\frac{1}{12}\frac{\sin u}{\cos u}$

$\therefore$ g (u) = -$\frac{1}{12}$ tan u

$\therefore$g′ (u) = -$\frac{1}{12}$ sec² u

By above result,

$x^2\frac{\partial^2 u}{\partial^2 x}$ + 2xy $\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y}$ = -$\frac{1}{12}$ [-$\frac{1}{12}$sec² u - 1]

= $\frac{1}{12}$ [$\frac{1}{12}$sec² u + 1] = $\frac{1}{12}$ [$\frac{1+tan^2 u}{12}$ + 1]

= $\frac{1}{144}$ tan u [tan² u + 13]

$\therefore x^2\frac{\partial^2 u}{\partial^2 x}$ + 2xy $\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y} = \frac{\tan u}{144}$ [tan ²u + 13].

Hence proved

(c) Expand 2 x^3 + 7 x^2 + x – 6 in power of (x – 2) by using Taylors Theorem. [4]

ANS: By Taylor's series,

f(x) = f(a) + (x +a)f′(a) + $\frac{(x+a)^2}{2!}$ f′′(a) + $\frac{(x+a)^3}{3!}$ f′′′(a) + ……

muquestionpapers.com

Here,

$f(x) = 2x^3 + 7x^2 + x - 6$

$f(2) = 2(2)^3 + 7(2)^2 + 2 - 6 = 40$

$f'(x) = 6x^2 + 14x + 1$

$f'(2) = 6(2)^2 + 14(2) + 1 = 53$

$f''(x) = 12x + 14$

$f''(2) = 12(2) + 14 = 38$

$f'''(x) = 12$

$f'''(2) = 12$

$f''''(x) = 0$.

$f(x) = f(2) + (x - 2)f'(2) + \frac{(x-2)^2}{2!} f''(a) + \frac{(x-2)^3}{3!} f'''(2) + 0$

$2x^3 + 7x^2 + x - 6 = 40 + (x - 2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12)$

$2x^3 + 7x^2 + x - 6 = 2(x - 2)^3 + 19(x - 2)^2 + 53(x - 2) + 40$

(d) Expand sec x by McLaurin's theorem considering up to x^4 term. [4]

ANS: Let y = sec x

$y = 1/(\cos x)$

$y = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots}$

$y = (1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots)^{-1}$

$y = 1 - (\frac{-x^2}{2} + \frac{x^4}{24}) + (\frac{-x^2}{2} + \ \dots)^2 + \dots$

$y = 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \dots\dots$

$\therefore$ **$y = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots\dots$**

muquestionpapers.com