APPLIED MATHEMATICS 1

(CBCGS 2016)

Q1]a) If cosαcosβ = $\frac{x}{2}$, sinαsinβ$\frac{y}{2}$, prove that :- **(3)**

$$\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{4x}{x^2-y^2}$$

Solution:- cosαcosβ = $\frac{x}{2}$ and sinαsinβ= $\frac{y}{2}$ ....(given)

$$\sec(\alpha - i\beta) = \frac{1}{\cos(\alpha-i\beta)} = \frac{1}{\frac{\cos\alpha\cos i\beta+\sin\alpha\sin i\beta}{\cos\alpha\cosh\beta+i\sin\alpha\sinh\beta}} \quad \frac{1}{\frac{x}{2}+\frac{iy}{2}} = \frac{2}{x+iy} \quad \text{......................(1)}$$

similarly for sec(α + iβ) we get ,

$$\sec(\alpha + i\beta) = \frac{2}{x-iy} \quad \text{..................(2)}$$

from (1) and (2)

$$\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{2}{x+iy} + \frac{2}{x-iy} = \frac{4x}{x^2-y^2}$$

Q1]b) If Z = log(e^x + e^y) show that rt - s^2 = 0 where r = $\frac{\partial^2 Z}{\partial x^2}$, t = $\frac{\partial^2 Z}{\partial y^2}$ s = $\frac{\partial^2 Z}{\partial x\,\partial y}$

Solution :- **(3)**

Z = log(e^x + e^y)

(1) $$\frac{\partial z}{\partial x} = \frac{e^x}{(e^x+e^y)} \qquad \frac{\partial^2 Z}{\partial x^2} = \frac{e^x(e^x+e^y)-e^x(e^x)}{(e^x+e^y)^2} = \frac{e^{2x}+e^{xy}-e^{2x}}{(e^x+e^y)^2}$$

$$r = \frac{\partial^2 Z}{\partial x^2} = \frac{e^{xy}}{(e^x+e^y)^2} \quad \text{.............(1)}$$

(2) $\frac{\partial z}{\partial y} = \frac{e^y}{(e^x + e^y)}$ $\quad\quad$ $\frac{\partial^2 Z}{\partial y^2} = \frac{e^y(e^x+e^y)-e^y(e^y)}{(e^x+e^y)^2} = \frac{e^{2y}+e^{xy}-e^{2y}}{(e^x+e^y)^2}$

$t = \frac{\partial^2 Z}{\partial y^2} = \frac{e^{xy}}{(e^x+e^y)^2}$ ....(2)

(3) $\frac{\partial z}{\partial x} = \frac{e^x}{(e^x+e^y)}$ $\quad\quad$ $s = \frac{\partial^2 Z}{\partial x\,\partial y} = \frac{e^{xy}}{(e^x+e^y)^2}$ ....(3)

From (1) , (2) and (3) we get ,

$rt = (\frac{e^{xy}}{(e^x+e^y)^2}) \times (\frac{e^{xy}}{(e^x+e^y)^2}) = \left(\frac{e^{xy}}{(e^x+e^y)^2}\right)^2 = (\frac{e^{2xy}}{(e^x+e^y)^2})$ ....(4)

$s^2 = \left(\frac{e^{xy}}{(e^x+e^y)^2}\right)^2 = (\frac{e^{2xy}}{(e^x+e^y)^2})$ ....(5)

From (4) and (5) we get,

$rt - s^2 = (\frac{e^{2xy}}{(e^x+e^y)^2}) - (\frac{e^{2xy}}{(e^x+e^y)^2}) = 0.$

Hence proved $rt - s^2 = 0$

Q1] c) If x=uv , y = $\frac{u+v}{u-v}$ **. find** $\frac{\partial(u,v)}{\partial(x,y)}$. **(3)**

Solution:- $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$x = uv, \quad y = \frac{u+v}{u-v}$ ....(given)

we know that JJ' = 1 ....(1)

the equation can also be solved by this following method.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = \partial(uv) = v.$ .... (2)

$\frac{\partial x}{\partial v} = \partial(uv) = u.$ ....(3)

$$\frac{\partial y}{\partial u} = \partial\left(\frac{u+v}{u-v}\right) = \frac{(u-v)-(u+v)}{(u-v)^2} \quad u-v-u+v/u-v\text{^}2 \quad = \frac{-2v}{(u-v)^2} \qquad \text{....................(4)}$$

$$\frac{\partial y}{\partial v} = \partial\left(\frac{u+v}{u-v}\right) = \frac{(u-v)+(u+v)}{(u-v)^2} = \frac{2u}{(u-v)^2} \qquad \text{....................(5)}$$

From equation (2), (3), (4), (5) we get,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix} = \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}.$$

From (1) we get,

JJ' = 1

$$J \times \frac{4uv}{(u-v)^2} = 1 \quad \text{..................(let } J' = \frac{4uv}{(u-v)^2}\text{)}$$

Hence $J = \frac{(u-v)^2}{4uv}$.

$$\therefore e^{2\varphi} = \cot\frac{\alpha}{2}$$

Q1] d) If y = $2^x \sin^2$**xcosxfind** y_n **(3)**

Solution :- $2^x = e^{x\log 2} = e^{ax}$ where a = log2

$$\frac{2\sin^2 x\cos x}{2} = \frac{\sin^1 x\cos x.\sin x \times 2}{2} = \frac{\sin x.\sin 2x}{2} = \frac{2\sin x.\sin 2x}{2\times 2} = \frac{\cos x}{4} - \frac{\cos 3x}{4}$$

$$\therefore \sin^2 x\cos x = \frac{\cos x}{4} - \frac{\cos 3x}{4}$$

$$Y = \frac{e^{ax}\cos x}{4} - \frac{e^{ax}\cos 3x}{4}$$

$$y_n = \tfrac{1}{4} r_1^n e^{ax} \cos\left(x+n\varphi_1\right) - \tfrac{1}{4} r_2^n e^{ax} \cos\left(3x+n\varphi_2\right)$$

$$y_n = \tfrac{1}{4} r_1^n 2^{1x} \cos\left(x+n\varphi_1\right) - \tfrac{1}{4} r_2^n 2^{1x} \cos\left(3x+n\varphi_2\right)$$

$$r_1 = \sqrt{(\log 2)^2} + 1 \qquad r_2 = \sqrt{(\log 2)^2} + 3^2$$

$$\varphi_1 = \tan^{-1}\left[\frac{1}{\log 2}\right] \qquad \varphi_2 = \tan^{-1}\left[\frac{3}{\log 2}\right]$$

Q1]e) Express the matrix as the sum of symmetric and skew symmetric matrices.

Solution:- **(4)**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} \qquad \mathbf{A'} = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A+A') = \frac{1}{2}\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A-A') = \frac{1}{2}\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$$

Let $P = \frac{1}{2}(A+A') = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$

$P' = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$

Hence P = P'. P is a symmetric matrix.

Let $Q = \frac{1}{2}(A-A') = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$

$$Q' = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$$

Hence Q = Q'. Q is a skew symmetric matrix.

Q1] f) Evaluate $\lim_{x\to 0} \frac{e^{2x} - (1+x)^2}{x\log(1+x)}$ **(4)**

Solution :- $\lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{\frac{x\log(1+x)}{x}.x} = \lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{x.x}$

$$= \lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{x^2}$$

Applying L-Hospital rule

$$\lim_{x\to 0} 1. \frac{2e^{2x} - 2(1+x)^1}{2x} = \lim_{x\to 0} 1. \frac{4e^{2x} - 2}{2} = \frac{4e^0 - 2}{2} = 1.$$

Q2]a) Show that the roots of x^5=1 can be written as 1,α^1,α^2,α^3,α^4 . hence show that (1 -α^1)(1-α^2)(1-α^3)(1-α^4) = 5. (6)

Solution:- $x^5 = 1 = \cos 0 + i\sin 0$

$\therefore x^5 = \cos(2k\pi) + i\sin(2\,k\pi)$

$\therefore x^1 = \left(\cos(2k\pi) + i\sin(2\,k\pi)\right)^{\frac{1}{5}} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$

Putting k =0,1,2,3,4 we get the five roots as

$x_0 = \cos 0 + i\sin 0 = 1$, $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, $x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$,

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$, $x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$.

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$ we see that $x_2 = \alpha^2$, $x_3 = \alpha^3$, $x_4 = \alpha^4$

$\therefore$ the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ and hence

$\therefore\ x^5-1 = (x-1)(x-\alpha\alpha(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore\ \frac{x^5-1}{(x-1)} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore\ (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4+x^3+x^2+x^1+1.$

Putting x=1, we get

$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$

Q2]b) Reduce the following matrix to its normal form and hence find its rank.

Solution:- **(6)**

$$A = \begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$2R_3 - R_1$ $\qquad$ $R_1 - R_3$ $\qquad$ $R_1 - R_2$ $\qquad$ $R_2 - R_4$

$$\begin{bmatrix} 1 & 2 & 6 & -3 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 6 & -3 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$R_2 - R_4$ $\qquad$ $4R_2 - R_3$ $\qquad$ $R_1 + 2R_4$ $\qquad$ $R_3/(-9)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 4 & 9 & -5 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -9 & 29 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -9 & 29 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & -2 & 5 \end{bmatrix}$$

$2R_3 + R_4$ $\qquad$ $6C_2 - C_4$ $\qquad$ $29/9C_3 + C_4$ $\qquad$ $R_4\ /(-13/9)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence the given matrix is converted to its normal form

Q2] c) Solve the following equation by Gauss-Seidel method up to four iterations

4x-2y-z = 40, x-6y+2y = -28, x-2y+12z = -86. (8)

Solution:- we first write the equation as

$x = ¼\,[40+2y+z]$ ……………………….(1)

$y = \frac{1}{6}[28+x+2z]$ …………………….(2)

$z = \frac{1}{12}[-86-x+2y]$ ……………………..(3)

(i) FIRST ITERATION :-

we start with the approximation y=0 , z=0 and then we get from (1),

$\therefore\ x_1 = ¼\,(40) = 10$

We use this approximation to find y i.e. put x=0, z=0 in (2)

$\therefore\ y_1 = \frac{1}{6}[28+10+2(0)] = 6.3333$

We use these values of x_1 and y_1 to find z_1 i.e. we put x =4 , y = 6.3333 in (3),

$\therefore\ z_1 = \frac{1}{12}[-86-10+2(6.3333)] = -6.944$

(ii) SECOND ITERATION :-

We use latest values of y and z to find x i.e. we put y =6.3333 , z = -6.9444 in (1)

$\therefore\ x_2 = ¼\,[40+2(6.3333)-6.9444] = 11.4306$

We use this approximation to find y i.e. put x=11.4306, z=-6.9444 in (2)

$\therefore\ y_2 = \frac{1}{6}[28+11.4306+2(-6.9444)] = 4.2569$

i.e. we put x =11.4306 , y = 4.2569 in (3),

$\therefore\ z_1 = \frac{1}{12}[-86-11.4306+2(4.2569)] = -7.4097$

(iii) THIRD ITERATION :-

We use latest values of y and z to find x i.e. we put y =4.2569 , z = -7.40974 in (1)

$\therefore\ x_2 = ¼\,[40+2(4.2569)-7.4097] = 10.2760$

We use this approximation to find y i.e. put x=10.2760 , z = -7.4097 in (2)

$$\therefore y_2 \;=\; \frac{1}{6}[28+10.2760+2(-7.4097)] \;=\; 3.9094$$

i.e. we put x =10.2760 , y = 3.9094 in (3),

$$\therefore z_1 \;=\; \frac{1}{12}[-86-10.2760+2(3.9094)] \;=\; -7.3714.$$

(iv) FOURTH ITERATION:-

We use latest values of y and z to find x i.e. we put y =3.9094 , z = -7.3714 in (1)

$$\therefore x_2 \;=\; ¼\,[40+2(3.9094)-7.3714] \;=\; 10.1118$$

We use this approximation to find y i.e. put x=10.1118 , z = -7.3714 in (2)

$$\therefore y_2 \;=\; \frac{1}{6}[28+10.1118+2(-7.3714)] \;=\; 3.8948$$

i.e. we put x =10.1118 , y = 3.8448 in (3),

$$\therefore z_1 \;=\; \frac{1}{12}[-86-10.1118+2(3.8948)] \;=\; -7.3602.$$

Hence , upto two places of decimals

x = 10.11 , y = 3.89, z = -7.36.

Q3] a) Investigate for what values of μ and λ the equations $x+ y+ z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ has

1) **No solution**
2) **A unique solution**
3) **Infinite number of solutions.** **(6)**

Solution:- we have $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

By $R_2 - R_1$, $R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-10 \end{bmatrix}$$

i) The system has unique solution if the coefficient matrix is non-singular (or the rank A, r= the number of unknowns , n =3).

This requires $\lambda - 3$ not equal to 0,

Hence λ is not equal to 3.

Hence the system has unique solution.

ii) If $\lambda = 3$ the coefficient matrix and the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \mu-10 \end{bmatrix}$$

The rank of A = 2 the rank of [A,B] will be also 2 if $\mu = 10$.

Thus if $\lambda = 33$and $\mu = 10$ the system is consistent. But the rank of A (= 2) is less than the number of unknowns (=3). Hence the equation will posses infinite solutions.

iii) If $\lambda = 3$ and $\mu \neq 10$, the rank of A=2, and the rank of [A,B] = 3. They are not equal and the equations will be inconsistent and will not posses any solution.

Q3]b) If $u = x^2 + y^2 + z^2$ where $x = e^t$, $y = e^t\sin t$, $z = e^t\cos t$

Prove that $\frac{du}{dt} = 4e^{2t}$. **(6)**

Solution:-

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$\frac{\partial u}{\partial x} = 2x \ , \ \frac{\partial u}{\partial y} = 2y \ , \ \frac{\partial u}{\partial z} = 2z$$

$$\frac{dx}{dt} = e^t \ , \ \frac{dy}{dt} = e^t(\sin t+\cos t) \ , \ \frac{dz}{dt} = e^t(-\sin t+\cos t)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$= 2x(e^t) + 2y(e^t(\sin t+\cos t)) + 2z(e^t(-\sin t+\cos t))$

$= 2x(x) + 2y(y+z) + 2z(z-y)$

$= 2x^2 + 2y^2 + 2z^2 + 2xy - 2xy$

$= 2x^2 + 2y^2 + 2z^2$

$= 2(x^2 + y^2 + z^2)$

$= 2u$ ....(1)

$u = x^2 + y^2 + z^2 = ((e)^t)^2 + (e^t\sin t)^2 + (e^t\cos t)^2$

$= e^{2t} + e^{2t}\left(\sin^2 t + \cos^2 t\right)$

$= e^{2t} + e^{2t} \ = 2e^{2t}$

Substituting value of u in equation (1)

$\frac{du}{dt} = 2u = 2(2e^{2t}) = 4e^{2t}$

Hence proved

$\frac{Du}{dt} = 4e^{2t}.$

Q3]c) i) Show that $\sin(e^x - 1) = x^1 + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots\ldots\ldots$ (4)

Solution :- We have $\sin(e^x-1) = \sin\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots\ldots\ldots\ldots -1\right)$

$\therefore \ \sin(e^x-1) = \sin\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ \ldots\ldots\ldots\right)$

But $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

$\therefore \ \sin(e^x-1) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots - \frac{1}{6}\left(x + \frac{x^2}{2} + \ldots\ldots\right)^3 + \ldots\ldots$

$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots - \frac{x^3}{6} - \frac{x^4}{4} + \ldots\ldots$

$= x + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots\ldots\ldots$

Q3]c) ii) Expand $2x^3 + 7x^2 + x - 6$ in powers of (x-2) (4)

Solution :- Let $f(x) = 2x^3 + 7x^2 + x - 6$ and $a=2$

$\therefore f'(x) = 6x^2 + 14x + 1$, $f''(x) = 12x + 14$, $f'''(x) = 12$

$\therefore f(2) = 45$, $f'(2) = 53$, $f''(2) = 38$, $f'''(2) = 12.$

Now , $f(x) = f(a) + f(x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \ldots\ldots\ldots\ldots$

$2x^3 + 7x^2 + x - 6 = 45 + (x-2).53 + (x-2)^{2.19} + (x-2)^{3.2}$

Q4] a) If $x = u+v+w$, $y = uv+vw+uw$, $z = uvw$ and φ is a function of x,y and z.

Prove that $x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z} = u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w}$ **(6)**

Solution:- φ is a function of x,y and z and x,y,z are themselves functions of u,v,w.

$$\therefore \frac{\partial\varphi}{\partial u} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial u} = \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(v+w) + \frac{\partial\varphi}{\partial z}vw$$

And $$\frac{\partial\varphi}{\partial v} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial v} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial v}$$

$$= \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(u+w) + \frac{\partial\varphi}{\partial z}.uw$$

And $$\frac{\partial\varphi}{\partial w} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial w} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial w} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial w}$$

$$= \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(v+u) + \frac{\partial\varphi}{\partial z}.uv$$

Multiplying (1) by u , (2) by v, (3) by w and add

$$\therefore u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w} = (u+v+w)\frac{\partial\varphi}{\partial x} + [(uv+uw) + (vu+vw)+(wv + wu)]\frac{\partial\varphi}{\partial y} + 3uvw\frac{\partial\varphi}{\partial z}$$

$$= (u+v+w)\frac{\partial\varphi}{\partial x} + [2(uv+vw+uw)]\frac{\partial\varphi}{\partial y} + 3uvw\frac{\partial\varphi}{\partial z}$$

$$= x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z}$$

$$\therefore x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z} = u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w}$$

Q4] b) If $\tan(\theta + i\varphi) = \tan\alpha + i\sec\alpha$

Prove that **1)** $e^{2\varphi} = \cot\frac{\alpha}{2}$ **2)** $2\theta = n\pi + \frac{\pi}{2} + \alpha.$ **(6)**

Solution :- $\tan(\theta + i\varphi) = \tan\alpha + i\sec\alpha$ $\therefore \tan(\theta - i\varphi) = \tan\alpha - i\sec\alpha$

$\therefore\ \tan 2\theta = \tan[(\theta + i\varphi i\varphi + (\theta - i\varphi)] \quad = \quad \dfrac{\tan(\theta+i\varphi)+\tan(\theta-i\varphi)}{1-\tan(\theta+i\varphi)\tan(\theta-i\varphi)}$

$$= \frac{\tan(\theta+i\sec\alpha)+\tan(\theta-i\sec\alpha)}{1-\tan(\theta+i\sec\alpha)\tan(\theta-i\sec\alpha)} \quad = \quad \frac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)} = \frac{2\tan\alpha}{-2\tan^2\alpha}-\cot\alpha \quad = \tan(\frac{\pi}{2}+\alpha)$$

$\therefore\ 2\theta = n\pi + \dfrac{\pi}{2} + \alpha.$ (general value).

Again $\tan(2i\varphi) = \tan[(\theta + i\varphi) - (\theta - i\varphi)]$

$$= \frac{\tan(\theta+i\sec\alpha)-\tan(\theta-i\sec\alpha)}{1+\tan(\theta+i\sec\alpha)\tan(\theta-i\sec\alpha)}$$

$\therefore\ i\tanh 2\varphi = \dfrac{2i\sec\alpha}{2\sec^2\alpha} = i\cos\alpha \qquad \therefore \quad \tanh 2\varphi = \cos\alpha$

$$\therefore\ 2\varphi = \tanh^{\wedge}(-1)(\cos\alpha)(\cos\alpha\frac{1}{2}\log\left[\frac{1+\cos\alpha}{1-\cos\alpha}\right] \qquad = \frac{1}{2}\log\left[\frac{2\cos^2\left(\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right)}\right]\text{ogcot}\frac{\alpha}{2}$$

Q 4]c) Find the roots of the equation $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 2.1 correct to 3 places of decimals using Regula Falsi method.

Solution:- **(8)**

Given that a=2 and b =2.1.

$f(2) = (2)^4 + (2)^3 - 7(2)^2 - 2 + 5 \quad = \quad -1.$

$f(2.1) = (2.1)^4 + (2.1)^3 - 7(2.1)^2 - (2.1) + 5 \quad = \quad 0.739100.$

$x_1 = \dfrac{af(b)-bf(a)}{f(b)-f(a)} \quad = (2\times 0.73910—1))\times(2.1)0.739100—10.739100—1.05750.$ ....(1)

$f((x_1) = (2.05750)^4 + (2.05750)^3 - 7(2.05750)^2 - 2.05750 + 5$

$= -0.05973.$

$x_2 = \dfrac{af(x_1)-x_1f(a)}{f(x_1)-f(a)} \quad = (2\times(-0.05973)—1))\times(2.05750)0.05973—1.05750)0.05973—12$

...............(2)

$f((x_2) = (2.061152)^4 + (2.061152)^3 - 7(2.061152)^2 - 2.061152 + 5$

$= 0.005326.$

$x_3 = \dfrac{af(x_2)-x_2f(a)}{f(x_2)-f(a)} \quad = (2\times(0.005326)—1))\times(2.061152)0.005326—1.061152)0.005326—1.$

………………(3)

$f((x_3) = (2.06082)^4 + (2.06082)^3 - 7(2.06082)^2 - 2.06082 + 5$

$= -0.000582.$

$$x_4 = \frac{af(x_3) - x_3 f(a)}{f(x_3) - f(a)} = (2\times(-0.000582)—1))\times(2.06082)0.000582—12.06082)0.000582—1$$

………………(4)

Hence from (4) and (3) iteration we get that value of x is coinciding.

Therefore the final value of x is 2.0608.

Q5] a) If $y = (x + \sqrt{x^2 - 1})^m$, Prove that

$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$ (6)

Solution:-

$y = (x + \sqrt{x^2 - 1})^m$

taking + sign before the radical

$\therefore y_1 = m[(x + \sqrt{x^2 - 1})^{m-1}].[1+\frac{x}{\sqrt{x^2-1}}]$

$= m(x + \sqrt{x^2 - 1})^m).\frac{x}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$

$\sqrt{x^2 - 1} . y_1 = my$

Differentiating again w.r.t x,

$\sqrt{x^2 - 1} . y_2 + \frac{x}{\sqrt{x^2-1}} y_1 = m\, y_1$

$(x^2 - 1)\, y_2 + xy_1 = m\sqrt{x^2 - 1}.\, y_1 = m.my = my^2$

$(x^2 - 1)\, y_2 + xy_1 - my^2 = 0$

Hence after applying lebnitz's theorem we get,

$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Q5]b) Using the encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ encode and decode the message

I*LOVE*MUMBAI.

Solution:-

A B C D E F G H I J K L M N O P Q R S T U V W X

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

* Y. Z

27. 25. 26

I * L O V E * M U M B A I *

9 27 12 15 22 5 27 13 21 13 2 1 9 27

Encoding the message includes the following process.

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}9 & 12 & 22 & 27 & 12 & 2 & 9\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}9+27 & 12+15 & 22+5 & 27+13 & 12+13 & 2+1 & 9+27\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}36 & 27 & 27 & 40 & 25 & 3 & 36\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

Hence the encoded message we get as ,

36, 27, 27, 15, 27, 5, 40, 13, 25, 13, 3, 1, 36, 27.

Now the process of decoding is as follows.

Inverse of $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$

$$\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}36 & 27 & 27 & 40 & 25 & 3 & 36\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}36-27 & 27-15 & 27-5 & 40-13 & 25-13 & 3-1 & 36-27\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}9 & 12 & 22 & 27 & 12 & 2 & 9\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

Hence the original message is obtained again after encoding and decoding.

Q5]c) i) Considering only principal values separate into real and imaginary parts

$i^{\log(i+1)}$. **(4)**

Solution:- let $Z = i^{\log(i+1)}$ $\therefore$ logZ = log(1+i).logi

But $\log(i+1) = \log\sqrt{2} + i\tan^{-1} 1$ $= \log\sqrt{2} + i\frac{\pi}{4}$ and $\log i = i.\frac{\pi}{2}$

$\therefore$ $\log Z = (\log\sqrt{2} + i\frac{\pi}{4}).\ i.\frac{\pi}{2}$ $= [\frac{1}{2}\log 2 + i\frac{\pi}{4}\pi)/2$ $= \frac{-\pi^2}{8} + i\frac{\pi}{4}\log 2$ $= e^{-\frac{\pi^2}{8}+i\theta}$ $= e^{-\frac{\pi^2}{8}.i\theta}$

where $\theta = \frac{\pi}{4}\log 2$ $= e^{-\frac{\pi^2}{8}}[\cos\theta + i\sin\theta i\sin\theta$

$\therefore$ Real part of Z $= e^{-\frac{\pi^2}{8}}\cos\theta = e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$ $\therefore$nary part of Z $= e^{-\frac{\pi^2}{8}}\sin\left(\frac{\pi}{4}\log 2\right)$

Q5]c) ii) Show that $i\log(\frac{x-i}{x+i}) = \pi - 2\tan^{-1} x$ **(4)**

Solution:- we have $\log(x+i) = ½\log(x^2+1) + i\tan^{-1}\frac{1}{x}$

and $\log(x-i) = ½\log(x^2+1) - i\tan^{-1}\frac{1}{x}$

$\log(\frac{x-i}{x+i}) = \log(x-i) - \log(x+i)$

$= -2i\tan^{-1}\frac{1}{x}$ $= -2i(\frac{\pi}{2} - \tan^{-1} x)$

$\therefore$ $\log(\frac{x-i}{x+i}) = -i(\pi - 2\tan^{-1} x)$

$\therefore$ $i\log(\frac{x-i}{x+i}) = (\pi - 2\tan^{-1} x)$.

Q6]a) Using De Moivre's theorem prove that

$\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)$ **(6)**

Solution:- Let as above $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$(2\cos\theta)^6 = (x + \frac{1}{x})^6$

$= x^6 + 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} + 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} + 6x^1.\frac{1}{x^5} + \frac{1}{x^6}$

$= x^6 + 6x^5 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}$ ……………………..(1)

$(2i\sin\theta)^6 = (x - \frac{1}{x})^6$

$= x^6 - 6x^5 + 15x^2 - 20 + 15\frac{1}{x^2} - 6\frac{1}{x^4} + \frac{1}{x^6}$ ……………………..(2)

$(2\sin\theta)^6 = x^6 + 6x^5 - 15x^2 + 20 - 15\frac{1}{x^2} + 6\frac{1}{x^4} - \frac{1}{x^6}$

Subtracting (2) from (1),

$2^6(\cos^6\theta - \sin^6\theta) = [x^6 + 6x^5 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}] -$

$[- x^6 + 6x^5 - 15x^2 + 20 - 15\frac{1}{x^2} + 6\frac{1}{x^4} - \frac{1}{x^6}]$

$= 2(x^6 + \frac{1}{x^6}) + 15(x^2 + \frac{1}{x^2})$

$= 2\cos6\theta + 15\cos2\theta$ ………………….[$(x^6 + \frac{1}{x^6}) = \cos6\theta$]

$\therefore 2^6 (\cos^6\theta - \sin^6\theta) = 2\cos6\theta + 15\cos2\theta.$

$\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos6\theta+15\cos2\theta)$

Q6] b) If $u = \sin^{-1}(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}})^{1/2}$, Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(\tan^2 u + 13)$$ **(6)**

Solution:-

$Z = \sin u = \sqrt{(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}})} = f(u) = F(X,Y)$ say .

Putting X = xt, Y = yt

$$F(X,Y) = \sqrt{\left(\frac{X^{\frac{1}{3}} + Y^{\frac{1}{3}}}{X^{\frac{1}{2}} - Y^{\frac{1}{2}}}\right)} = \sqrt{\left(\frac{(xt)^{\frac{1}{3}} + (yt)^{\frac{1}{3}}}{(xt)^{\frac{1}{2}} - (yt)^{\frac{1}{2}}}\right)} = \sqrt{\frac{t^{1/3}}{t^{1/2}}\left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)} = t^{-1/12}f(x,y)$$

Thus Z = f(u) = sinu is a homogenous function of x, y of degrees 1/12

Hence , by the above corollary.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

$$\text{Where , } g(u) = n\frac{f(u)}{f'(u)} = \frac{-1}{12}.\frac{\sin u}{\cos u} = \frac{-1}{12}\tan u$$

$$g'(u)-1 = \frac{-1}{12}\sec^2 u - 1 = \frac{-1}{12}(1-\tan^2 u) - 1 = \frac{-1}{12}\tan^2 u - \frac{13}{12}$$

$$= \frac{-1}{12}(\tan^2 u + 13)$$

$$\therefore\ g(u)[g'(u)-1] = \left(\frac{-1}{12}\tan u\right)\left(\frac{-1}{12}(\tan^2 u + 13)\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(\tan^2 u + 13)$$

Q6]c) Find the maxima and minima of $x^3y^2(1-x-y)$ **(8)**

Solution :- we have $f(x) = x^3y^2(1-x-y)$

Step 1 :- $f_x = y^2[3x^2(1-x-y) - x^3] = y^2(3x^2-4x^3-3x^2y)$

$= (3x^2y^2-4x^3y^2-3x^2y^3)$

f_y = x^3 [2y(1 - x - y)y(1-x-y)y^2] $= x^3(2y-2xy-3y^2)$

$= (2yx^3-2x^4y-3x^3y^2)$

$\therefore\ f_{xx} = 6y^2x^1 - 12x^2y^2 - 6x^1y^3$

$\therefore\ f_{xy} = 6y^1x^2 - 8x^3y^1 - 9x^2y^2$

$\therefore\ f_{yy} = 2x^3 - 2x^4 - 6x^3$

Step 2:- we now solve for $f_y = 0$, $f_x = 0$

$\therefore 3y^2x^2 - 4x^3y^2 - 3x^2y^3 = 0$ i.e. $y^2x^2(3-4x-3y) = 0$

And $2y^1x^3 - 2x^4y^1 - 3x^3y^2 = 0$ i.e. $y^1x^3(2-2x-3y) = 0$

$\therefore$ x= 0 , y = 0 and $(3\text{-}4x\text{-}3y) = 0$, $2 - 2x - 3y3y\ 0$

Subtracting we get 1-2x = 0

$\therefore$ x = ½ $\therefore$ 3y = 3-4(1/2) = 1 $\therefore$ y = $\frac{1}{3}$

$\therefore$ (0,0) and (1/2 , 1/3) are stationary points.

Step 3:- at x = 0, y = 0 , r = 0 , s = 0 , t = 0 $\therefore$ rt - s^2 = 0

At x = ½ , y =1/3

r = f_{xx} = $6(1/2)(1/9) - 12(1/4)(1/9) - 6(1/2)(1/27)/9)\text{-}12(1/4)(1/9)\text{-}6(1/2)(1/27)\frac{1}{3} - \frac{1}{3} - \frac{1}{9}$

= $-\frac{1}{9}$

s = f_{xy} = $6(1/4)(1/3) - 8(1/8)(1/3) - 9(1/4)(1/9)3)\text{-}8(1/8)(1/3)\text{-}9(1/4)(1/9)$ = ½ - $\frac{1}{3}$ - ¼

= $-\frac{1}{12}$

t = f_{xy} = $2(1/8) - 2(1/16) - 6(1/8)(1/3)(1/16)\text{-}6(1/8)(1/3)$ = ¼ - $\frac{1}{8}$ - ¼ = $-\frac{1}{8}$

$\therefore$ rt - s^2 = $(-1/9)(-1/8) - (1/12)(1/12)9)(-1/8)\text{-}(1/12)(1/12)$ = $\frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$

And r = -1/9 <0 $\therefore$ f(x,y) is a maxima

Maximum value = $\frac{1}{8}.\frac{1}{9}\left(1\text{-}\frac{1}{2}\text{-}\frac{1}{3}\right) = \frac{1}{432}$
