Duration 2 $\frac{1}{2}$Hrs Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. (a) Attempt ANY ONE from the following: (8)
   (i) Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Show that f is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y.
   (ii) Let (X, d) and (Y, d') be metric spaces. If (X, d) is a compact metric space and $f : X \longrightarrow Y$ is a continuous function, then show that $f(X)$ is a compact subset of Y.

   (b) Attempt ANY TWO from the following: (12)
   (i) Let (X, d) be a metric space and $f, g : (X, d) \longrightarrow \mathbb{R}$ (usual distance) be continuous on X. Show that $f + g : (X, d) \longrightarrow \mathbb{R}$ is also continuous on X. Is the converse true?
   (ii) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions (with respect to usual distance). Let $h : (\mathbb{R}^2, d) \longrightarrow (\mathbb{R}^2, d)$ be defined by $h(x, y) = (f(x), g(y))$. Show that h is continuous on $(\mathbb{R}^2, d)$ where d is Euclidean distance.
   (iii) Prove or disprove:
   If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are uniformly continuous on a nonempty set $A \subseteq \mathbb{R}$ then the product function $f \cdot g$ is uniformly continuous on A.

2. (a) Attempt ANY ONE from the following: (8)
   (i) Define separated sets in a metric space, disconnected metric space. Prove that a metric space (X, d) is disconnected if and only if there exists a nonempty proper subset of X which is both open and closed in X.
   (ii) Define connected subset of a metric space. Prove that if a subset E of $\mathbb{R}$ is connected then it is an interval. (Distance in $\mathbb{R}$ being usual). Is $\mathbb{Q}$ connected? Justify.

   (b) Attempt ANY TWO from the following: (12)
   (i) Show that the set $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 2, 1 < y < 5\}$ is a convex set in $(\mathbb{R}^2, d)$ where d is the Euclidean distance.
   (ii) If (X, d) be a connected metric space and $f : X \longrightarrow \mathbb{Z}$ (distance in $\mathbb{Z}$ being usual distance) is a continuous function then prove that f is a constant function.
   (iii) Let (X, d) be a metric space and A, B be conncected subsets of X such that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is a connected set.

3. (a) Attempt ANY ONE from the following: (8)
   (i) Let $\{f_n\}$ be a sequence of real valued $R-$integrable functions defined on $[a, b]$ such that $f_n \longrightarrow f$ uniformly on $[a, b]$. Prove that f is $R-$ integrable on $[a, b]$ and $\lim_{n \longrightarrow \infty} \int_a^b f_n(t)\, dt = \int_a^b \lim_{n \longrightarrow \infty} f_n(t)\, dt$.

(ii) Prove that if the power series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x_1 \in \mathbb{R}, x_1 \neq 0$ and diverges at $x_2 \in \mathbb{R}$ then the power series $\sum_{n=0}^{\infty} |c_n x^n|$ converges for all $x \in \mathbb{R}$ with $|x| < |x_1|$ and diverges for all $x \in \mathbb{R}$ with $|x| > |x_2|$.

(b) Attempt ANY TWO from the following: (12)

(i) State and prove Cauchy Criterion for Uniform Convergence of a Series $\sum_{n=0}^{\infty} f_n$ of real valued functions defined on a subset S of $\mathbb{R}$.

(ii) Show that $\sum_{n=1}^{\infty} e^{-nx} x^n$ is uniformly convergent on $[0, A], A > 0$.

(iii) Let $f_n : [-1, 1] \longrightarrow \mathbb{R}$ with $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$. Given that $f_n \longrightarrow f$ uniformly on $[-1, 1]$ where $f(x) = |x|$ for $x \in [-1, 1]$. Find $\lim_{n \longrightarrow \infty} \int_{-1}^{1} f_n(x)\, dx$.

4. Attempt ANY THREE from the following: (15)

(a) Let (X, d) and (Y, d') be metric spaces. If $f : X \longrightarrow Y$ is uniformly continuous on X and (x_n) is a Cauchy sequence in X, then show that the sequence $(f(x_n))$ is Cauchy in Y.

(b) Define a contraction map. $T : \left[0, \frac{1}{3}\right] \longrightarrow \left[0, \frac{1}{3}\right]$ is defined as $T(x) = x^2$, show that T is a contraction map on $\left[0, \frac{1}{3}\right]$.

(c) Let (X, d) be a metric space. If A is a finite subset of X having more than one element, show that A is disconnected.

(d) Prove or disprove: The subset $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ of $(\mathbb{R}^2, d)$, d being Euclidean distance, is connected.

(e) Find the set of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$.

(f) For each $n \in \mathbb{N}$, let $f_n : [0, 1] \longrightarrow \mathbb{R}$ be defined by $f_n(x) = nxe^{-nx}$. Show that $\{f_n\}$ converges pointwise but not uniformly on $[0, 1]$.

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