Time: $2\frac{1}{2}$ hours **Total Marks:75**

Instructions:1) All questions are compulsory.
2) Figures to right indicate marks for respective sub questions.

Q.1 A		Answer any **ONE**	
	i)	Define a Field. Show that the ring $\mathbb{Z}_n$ of residue classes modulo n is a field if and only if n is a prime number.	08
	ii)	Define Integral domain and characteristic of ring. Show that characteristic of an integral domain is either zero or a prime number.	08
B		Answer any **TWO**	
	i)	Prove that in any ring R, p) $a.0 = 0$ for each $a \in R$. q) $a(-b) = -(ab)$ for every $a, b \in R$. r) $a(b-c) = ab - ac$ for all $a, b, c \in R$.	06
	ii)	Let R be an Integral domain of characteristic 2 then show that p) $(a+b)^2 = a^2 - b^2$ for each $a, b \in R$ q) $S = \{x \in R: x^2 = x\}$ is a subring of R.	06
	iii)	Define unit in ring. Let F be field then Show that the only units in F[X] are nonzero elements of F.	06
Q.2 A		Answer any **ONE**	
	i)	Let R be a commutative ring. If I, J are ideals of R. Show that $I + J = \{x + y \mid x \in I, y \in J\}$ and $IJ = \{\sum_{i=1}^{n} x_i y_i \mid x_i \in I, y_i \in J, n \in \mathbb{N}\}$ are ideals of R.	08
	ii)	Define Euclidean domain, Principal Ideal domain and prove that every Euclidean domain is Principal Ideal domain.	08
B		Attempt Any **TWO**	
	i)	Prove that the ring $\frac{\mathbb{Q}[x]}{<x^2-2>}$ is isomorphic to the ring $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$	06
	ii)	Show that an ideal M in a commutative ring R is maximal ideal if and only if R/M is a field.	06
	iii)	Show that an ideal I in the ring of integers $\mathbb{Z}$ is a prime ideal if and only if $I = (0)$ *or* $I = p\mathbb{Z}$, where p is a prime number.	06
Q.3 A		Answer any **ONE**	
	i)	Prove that for any pair of non- constant polynomials $f(X), g(X) \in F[X]$, there exists $q(X), r(X) \in F[X]$ such that $f(X) = g(X)q(X) + r(X)$, where $r(X) = 0$ or $\deg(r(X)) < \deg((g(X))$.	08

ii) Let R be an unique factorization domain and $a \in R$ then prove that ‘a’ is irreducible if and only if ‘a’ is prime element. 08

B Attempt Any **TWO**

i) Let R be an integral domain and $a \in R, a \neq 0_{\{R\}}$. If (a) is a maximal ideal then prove that a is irreducible. Give an example to show that the converse is not true 06

ii) List all monic polynomials of degree 2 over $\mathbb{Z}_3$. Which of these are irreducible? 06

iii) Let D be an Euclidean domain and let d be the associated function. Prove that $u \in D$ is a unit if and only if $d(u) = d(1)$. 06

Q.4 Answer any **THREE**

i) Find all zero divisors, units, idempotents and nilpotent elements of the ring $(\mathbb{Z}_{18}, +, .)$. 05

ii) Let R be an integral domain and $a, b \in R$. If $a^3 = b^3$ and $a^8 = b^8$ then show that $a = b$. Does this hold if $a^m = b^m, a^n = b^n$ and the greatest common divisor of m and n is 1? 05

iii) Let R be a commutative ring with prime characteristic p and $f: R \to R$ be defined as $f(a) = a^p$ for $a \in R$. Show that f is a ring homomorphism. 05

iv) Let $S = \{a + bi : a \in \mathbb{Z}, b \in 3\mathbb{Z}\}$. Show that S is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$. 05

v) Consider the ring $\mathbb{Z}[\sqrt{d}]$ where d is not 1 and is not divisible by square of any number. Define $N: \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}^{+} \cup \{0\}$ as $N(a + b\sqrt{d}) = |a^2 - db^2|$. Show that x is irreducible if $N(x)$ is prime. 05

vi) Using Division Algorithm find the quotient and remainder when $f(x) = 2x^3 + x^2 + 3x + 4$ is divided by $2x^2 + 1$ over $\mathbb{Z}_5$. 05

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