Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. For a simple graph G of order p and size q, prove that $\pi_k(G)$, the chromatic polynomial of the graph G, is monic polynomial of degree p in k with integer coefficients and constant term zero. Further prove that its coefficients are alternate in sign and the coefficient of k^{p-1} is $-q$.
   ii. If $G = K_n$ is complete graph with n vertices, $n \geq 2$ then prove that an edge chromatic number $\chi'(G) = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd} \end{cases}$

   (b) Attempt any **TWO** questions: (12)
   i. Define a chromatic polynomial of graph G. Determine the chromatic polynomial and chromatic number of a graph G obtained by deleting an edge from K_4.
   ii. If G is a cycle on n vertices then show that $\pi_k(G) = (k-1)^n + (-1)^n(k-1)$.
   iii. Show that vertex connectivity of a graph G is always less or equal to the edge connectivity of G.

2. (a) Attempt any **ONE** question: (8)
   i. Show that every planar graph is 5 vertex colorable.
   ii. Show that there are exactly five regular Polyhedra.

   (b) Attempt any **TWO** questions: (12)
   i. State and prove Euler theorem for planar graph.
   ii. Let G^* denote dual graph of G. Show that the edge e is a loop in G if and only if e^* is a bridge in G^*.
   iii. Let f be a flow in a network N and P be any f-incrementing path then show that there exist a revised flow f' such that $val(f') = val(f) + \epsilon(p)$

3. (a) Attempt any **ONE** question: (8)
   i. State and prove Hall's (Marriage) Theorem for a System of Distinct Representatives.
   ii. Derive the recurrence relation for the number of regions into which the plane is divided by n straight lines, no two of which are parallel and no three of which are concurrent. Furthermore using generating function, show that the solution of the above recurrence is $\frac{n(n+1)}{2} + 1$.

   (b) Attempt any **TWO** questions: (12)
   i. Let $R_{n,m}(x)$ be the rook polynomial for the $n \times m$ chess board, all squares may have rooks. Show that $R_{n,m}(x) = R_{n-1,m}(x) + mxR_{n-1,m-1}(x)$.

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ii. Find the number h_n of bags of fruit that can be made out of apples, bananas, oranges, and pears, where, in each bag, the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1.

iii. Find the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

with $1 \leq x_1 \leq 6, 0 \leq x_2 \leq 7, 4 \leq x_3 \leq 8, 2 \leq x_4 \leq 6$.

4. Attempt any **THREE** questions: (15)

(a) Define $k-$critical graph. If G is $k-$ critical graph then show that $\delta(G) \geq k-1$ where $\delta(G)$ is minimum degree of G.

(b) Show that $\chi'(G) \geq \Delta(G)$ where $\chi'(G)$ denotes edge chromatic number and $\Delta(G)$ denotes the maximum degree of G. Give an example of the graph for which $\chi'(G) = \Delta(G)$.

(c) Show that there is at least one face of every polyhedron is bounded by an $n-$cycle for some $n = 3, 4$ or 5.

(d) If f is any flow and K be any cut in a network N then show that $val(f) \leq cap(K)$.

(e) Solve recurrence relation $a_n = 3a_{n-1}, n \geq 1$ given $a_0 = 1$ by using generating function method.

(f) Find a recurrence relation for the ways to distribute n identical balls into k distinct boxes with between two and four balls in each box. Repeat the problem with balls of three colors.

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