Total Marks: 75 **Time: 2.5 Hours**

N. B. 1) All questions are compulsory.
2) Use of a simple calculator is allowed.
3) Figures to the right indicate marks.

Q. 1 A) Attempt any one from the following. (8)

i) Let $\Omega \subset \mathbb{C}$ is a domain in $\mathbb{C}$. If $u, v : \Omega \to \mathbb{R}$ are such that

i) u_x, u_y, v_x, v_y exist and satisfy Cauchy Riemann equations

ii) u_x, u_y, v_x, v_y are continuous on Ω,

then prove that $f(z) = u(x, y) + iv(x, y)$ is analytic in Ω.

ii) If z_0 and w_0 are points in z and w plans respectively then show that

(a) $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$.

(b) $\lim_{z \to \infty} f(z) = \omega_0$ if and only if $\lim_{z \to 0} f\left(\frac{1}{z}\right) = \omega_0$.

(c) $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$.

B Attempt any two from the following. (12)

i) If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then show that its component functions u and v are harmonic in D.

ii) Use $\epsilon - \delta$ definition of limit to show that $\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i$.

iii) Let f be a analytic function throughout on a given domain D. If $|f(z)|$ is constant on D, show that $f(z)$ must be constant on D.

Q.2 A) Attempt any one from the following. (8)

i) State and prove extension of Cauchy's Integral formula.

ii) Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0. Then prove that $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $|z - z_0| < R_0$ where $a_n = \frac{f^n(z_0)}{n!}$ i.e. the series converges to $f(z)$ when z lies in the stated open disk.

B Attempt any two from the following. (12)

i) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non negative constant such that $|f(z)| \le M \; \forall z \in C$ at which $f(z)$ is defined then prove that $\left|\int_C f(z)dz\right| \le ML$.

ii) Evaluate $\int_C \frac{\sin^6 z}{\left(z - \frac{\pi}{2}\right)^3} dz$, where $C : |z| = 2$.

iii) Find a linear fractional transformation that maps the points $1, i, -1$ onto the points $-1, 0, 1$ on the real axis.

Q.3 A) Attempt any one from the following. (8)

i) If a series $\sum a_n (z - z_0)^n$ converges to $f(z)$ at all points within the disc of convergence $|z - z_0| < R$ then prove that it is the Taylor series expansion for f centered at z_0.

ii) Let C be a simple closed curve in the interior of the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and let $g(z)$ be any function which is continuous on C. Then prove that the series $\sum_{n=0}^{\infty} g(z) a_n (z - z_0)^n$ can be integrated term by term over C and

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} \int_C g(z)a_n(z - z_0)^n dz.$$

B Attempt any two from the following. (12)

i) If the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for $z = z_1 (\neq z_0)$, then prove that it is absolutely convergent for each $z \in B(z_0, R_1)$ where $R_1 = |z_1 - z_0|$.

ii) If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R then find the radius of convergence of

$(a) \sum_{n=0}^{\infty} n^3 a_n z^n$ (b) $\sum_{n=0}^{\infty} a_n z^{3n}$ (c) $\sum_{n=0}^{\infty} {a_n}^3 z^n$.

iii) Find Laurent series expansions in the domains: $|z| < 1$, $1 < |z| < 2$, $2 < |z| < \infty$ for $f(z) = \frac{-1}{(z-1)(z-2)}$.

Q. 4 A Attempt any three questions from the following. (15)

i) Represent $|z - z_0| = |z - \bar{z}_0|$ as subsets of $\mathbb{C}$ in the plane where $Im\, z_0 \neq 0$.

ii) Show that $z(t) = z_0 + tv$ and $Re\big((z - z_0) i\, \bar{v}\big) = 0$ represents the same line in $\mathbb{C}$

iii) Find all roots of the equation $\cos z = 2$.

iv) Determine whether the set of points $0, -4, -2i, -1 - 3i$ lies on a circle.

v) Find residue of $f(z)$ at $z = 0$ where $f(z) = \frac{\cot z}{z^4}$ (using the idea of power series division).

vi) Using Cauchy Residue theorem, evaluate $\int_C f(z)dz$ where $f(z) = \frac{1}{(z-1)^2(z-3)}$ and C is bounded by $x = 0, x = 4, y = -1, y = 1$.
