Duration:[3 Hours] **[Total Marks: 100]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. Let G be a connected graph. If G is neither complete nor an odd cycle then vertex chromatic number of G is
   (a) $\leq$ Max degree of G. (b) $=$ Maximum degree of G.
   (c) $=$ Max degree of $G + 1$ (d) None of these

ii. Edge Chromatic number of $K_{1,n}, n \geq 2$ is
   (a) 2 (b) n
   (c) $n + 1$ (d) None of the above

iii. The relation between vertex connectivity κ, edge connectivity κ' and minimum degree δ is
   (a) $\kappa = \kappa' = \delta$ (b) $\kappa < \kappa' < \delta$
   (c) $\kappa \leq \kappa' \leq \delta$ (d) None of these

iv. If $G(p, q)$ be planar graph, then sum of degrees of regions is equal to
   (a) $2p$ (b) $2q$
   (c) $p + q$ (d) None of these

v. Let G be a connected planar graph with q edges, p vertices and f regions then
   (a) $2q > 3f$ (b) $2q \geq 3f$
   (c) $2q < 3f$ (d) $2q \leq 3f$

vi. If f is a flow in a network N and S be any subset of nodes, then $(S, \overline{S})$ is a cut if
   (a) $x \in S, y$ is not in $\overline{S}$ (b) x is not in S, $y \in \overline{S}$
   (c) x is not in S, y is not in $\overline{S}$ (d) $x \in S, y \in \overline{S}$

vii. If the polynomial $c + 3x + x^2$ is a rook polynomial of some chess board then
   (a) $c = 0$ (b) $c = 1$
   (c) $c > 1$ (d) c is even

viii. The function e^{-x} is the generating function of the sequence
   (a) $a_n = \frac{1}{n!}$ (b) $a_n = \frac{-1}{n!}$
   (c) $a_n = \frac{(-1)^n}{n!}$ (d) $a_n = (-1)^n$

ix. The number of different system of distinct representatives of the family $A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 4\}, A_4 = \{4, 5\}, A_5 = \{5, 1\}$
   (a) 1 (b) 2
   (c) 3 (d) 5

x. A matching M in G is a maximum matching if and only if G contains
   (a) no M-augmenting path. (b) M-augmenting path.
   (c) no M-alternating path. (d) None of these

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2. (a) Attempt any **ONE** question from the following: (8)

    i. If G is $k-$ critical graph then show that
    I) G is connected
    II) Every vertex v of graph G has at least $k-1$ degree.
    III) Graph G cannot be partitioned into subgraphs.

    ii. Prove that a graph G with $p \geq 2$ is 2-connected if and only if any two vertices are connected by at least two internally disjoint paths.

   (b) Attempt any **TWO** questions from the following: (12)

    i. Define vertex chromatic number $\chi(G)$. Let G be the graph with n vertices. Show that $\chi(G) \geq \frac{n}{n-\delta(G)}$ where $\chi(G)$ denotes vertex chromatic number of G and $\delta(G)$ denotes minimum degree of G.

    ii. If G is a cycle on n vertices then show that $\pi_k(G) = (k-1)^n + (-1)^n(k-1)$.

    iii. If G is cubic graph, then show that $\kappa(G) = \kappa'(G)$ where $\kappa(G)$ denote the vertex connectivity and $\kappa^{'}(G)$ denotes the edge connectivity of a graph G.

    iv. If G is a bipartite graph, then show that $\chi'(G) = \Delta(G)$, where $\chi(G)$ represents vertex chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G.

3. (a) Attempt any **ONE** question from the following: (8)

    i. State and prove Euler's formula for planar graphs. Hence or otherwise prove that the minimum degree of a simple planar graph is ≤ 5.

    ii. State and prove Max Flow - Min Cut Theorem.

   (b) Attempt any **TWO** questions from the following: (12)

    i. Define dual graph G^* of G. Show that edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.

    ii. Let f be a flow in a network N and P be any f-incrementing path then show that there exist a revised flow f' such that $val(f') = val(f) + \epsilon(p)$

    iii. For a plane graph G, prove that G is bipartite if and only if every face of G has even length.

    iv. Show that there is at least one face of every polyhedron is bounded by an $n-$cycle for some $n = 3, 4$ or 5.

4. (a) Attempt any **ONE** question from the following: (8)

    i. State and prove Hall's (Marriage) Theorem for a System of Distinct Representatives.

    ii. Derive the recurrence relation for number of ways of dividing a $n+1-$sided convex polygon into triangular regions by inserting diagonals that do not intersect in the interior and prove using generating function that the solution to this recurrence relation is a Catalan Number.

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(b) Attempt any **TWO** questions from the following: (12)

i. Let B denotes a forbidden chess board in which a special square $*$ has been identified and let D denote the board obtained from the original board by deleting the row and column containing the special square and E denote the board obtained from the original board where only the special square $*$ is removed from the board, then prove that $R(x, B) = xR(x, D) + R(x, E)$.

ii. Show that a matching M in G is a maximum matching if and only if G contains no $M-$augmenting path.

iii. Determine the generating function for the number of n-combinations of apples, bananas, oranges, and pears, where, in each n-combination, the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear.

iv. Solve the recurrence relation $a_n = 3a_{n-1} + 2$, with $a_0 = 1$ using generating function.

5. Attempt any **FOUR** questions from the following: (20)

(a) Show that vertex connectivity of a graph G is always less or equal to the edge connectivity of G.

(b) Show that if $G_1, G_2, \ldots, G_n$ are n components of graph G then $\pi_k(G) = \prod_{i=1}^{n} \pi_k(G_i)$.

(c) Show that every planar graph is 6-vertex colorable.

(d) If f is any flow and K be any cut in a network N with $val(f) = cap(K)$ then show that f is maximum flow and K is minimum cut.

(e) Show that the number of nonnegative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = r$ is given by $\begin{pmatrix} r+k-1 \\ r \end{pmatrix}$.

(f) Let $A = (A_1, A_2, A_3, A_4, A_5, A_6)$, where $A_1 = \{1, 2, 3\}, A_2 = \{1, 2, 3, 4, 5\}, A_3 = \{1, 2\}, A_4 = \{2, 3\}, A_5 = \{1\}, A_6 = \{1, 3, 5\}$. Does family A have an System of Distinct Representative? If not, what is the largest number of sets in the family with an System of Distinct Representative?

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