

Duration: 3 Hrs

Marks: 100

- N.B. : (1) All questions are compulsory.
 (2) Figures to the right indicate marks.

1. Choose the correct alternative for each of the following: (20)

- (i) If $f : (\mathbb{R}, d) \rightarrow (\mathbb{R}, d)$, with d as the usual distance, is a continuous function then $f^{-1}((0, \infty))$ is
- (a) a closed subset of \mathbb{R} . (c) an open subset of \mathbb{R}
 (b) a bounded subset of \mathbb{R} . (d) none of the above is true.
- (ii) Let (X, d) be a discrete metric space and (Y, d') be any metric space. If $f : X \rightarrow Y$, then f is
- (a) an uniformly continuous function on X .
 (b) a bounded function on X .
 (c) continuous but not uniformly continuous function on X .
 (d) none of the above is true.
- (iii) Consider the metrics d and d_1 on \mathbb{N} , where d is the induced distance from \mathbb{R} with the usual metric and $d_1(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|$ for $m, n \in \mathbb{N}$. Let $i : \mathbb{N} \rightarrow \mathbb{N}$ denote the identity map on \mathbb{N} . Then
- (a) $i : (\mathbb{N}, d) \rightarrow (\mathbb{N}, d_1)$ is continuous but $i : (\mathbb{N}, d_1) \rightarrow (\mathbb{N}, d)$ is not continuous.
 (b) $i : (\mathbb{N}, d) \rightarrow (\mathbb{N}, d_1)$ is not continuous.
 (c) $i : (\mathbb{N}, d_1) \rightarrow (\mathbb{N}, d)$ is not continuous.
 (d) None of the above.
- (iv) In (\mathbb{R}^2, d) where d is the Euclidean distance, the following set is not connected.
- (a) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$. (c) $\mathbb{R}^2 \setminus \{(x, y) : y = 0\}$
 (b) $\mathbb{R}^2 \setminus \{(0, 0)\}$ (d) None of the above.
- (v) Let A and B be connected subsets in a metric space (X, d) and $A \subseteq C \subseteq B$ Then,
- (a) C is connected. (c) \bar{C} is connected.
 (b) C° is connected. (d) $C \cap \bar{A}$ is connected.
- (vi) In \mathbb{R}^2 with the Euclidean metric, which of the following sets is convex?
- (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$ (c) $\{(x, y) \in \mathbb{R}^2 : y > 0\}$
 (b) $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ (d) None of these.
- (vii) Let (X, d) be a connected metric space and $f : X \rightarrow \mathbb{Z}$ be a continuous map. Then, f is
- (a) an onto function. (c) a bijective function.
 (b) a one-one function. (d) a constant function.

- (viii) Let $f_n(x) = \sin nx$ for $x \in \mathbb{R}$. and $g_n(x) = \frac{f_n(x)}{n} \quad \forall x \in \mathbb{R}$. Then,
- $\{f_n\}$ and $\{g_n\}$ are uniformly convergent on \mathbb{R} .
 - $\{f_n\}$ and $\{g_n\}$ are not pointwise convergent on \mathbb{R} .
 - $\{g_n\}$ is uniformly convergent on \mathbb{R} but $\{f_n\}$ is not.
 - $\{f_n\}$ is uniformly convergent on \mathbb{R} but $\{g_n\}$ is not.
- (ix) The series $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$ is
- uniformly convergent on \mathbb{R} .
 - not uniformly convergent on $[-a, a]$ where $0 < a < 1$
 - uniformly convergent on $[-a, a]$ where $0 < a < 1$.
 - none of the above.
- (x) If R is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ then the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} c_n x^n$ is
- R^2
 - R
 - 0
 - ∞

2. (a) Attempt any One of the following: (8)

- Let (X, d) and (Y, d') be metric spaces. If (X, d) is compact and $f : X \rightarrow Y$ is a continuous function, then show that $f(X)$ is a compact subset of Y .
- Let $f : (X, d) \rightarrow (Y, d')$ be a function. Prove that f is continuous on X if and only if for each open subset G of Y , $f^{-1}(G)$ is an open subset of X .

(b) Attempt any Two of the following: (12)

- Let (X, d) and (Y, d') be metric spaces then show that $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$, for each subset B of Y .
- (X, d) is a metric space and $f : (X, d) \rightarrow (X, d)$ is a function such that $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Let $S = \{x \in X : f(x) = x\}$. Prove that
 - f is continuous on X .
 - S has at most one element.
- Let (X, d) and (Y, d') be metric spaces. When is $f : X \rightarrow Y$ said to be uniformly continuous? Show that $f(x) = \frac{1}{(1+x^2)}$ is uniformly continuous on \mathbb{R} (under the usual metric).
- Let d_1 and d_2 be equivalent metrics on X and (Y, d) be any metric space. If $f : (X, d_1) \rightarrow (Y, d)$ and $g : (Y, d) \rightarrow (X, d_1)$ are continuous maps on X and Y respectively, then prove that $f : (X, d_2) \rightarrow (Y, d)$ and $g : (Y, d) \rightarrow (X, d_2)$ are also continuous.

3. (a) Attempt any One of the following: (8)
- (i) Define a connected metric space and prove that a metric space (X, d) is disconnected if and only if there exists a nonempty proper subset of X which is both open and closed in X .
 - (ii) Prove that a metric space is connected if and only if every continuous function from X to $\{1, -1\}$ is a constant function.
- (b) Attempt any Two of the following: (12)
- (i) Let (X, d) be a metric space such that for any $x, y \in X$ there exists a connected subset A of X such that $x, y \in A$. Prove that X is connected.
 - (ii) Prove that in a normed linear space, an open ball $B(x, r)$ is a convex set.
 - (iii) Prove that if a subset E of \mathbb{R} is connected then it is an interval. (Distance in \mathbb{R} being usual)
 - (iv) Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x = 1\}$ in (\mathbb{R}^2, d) where d is the Euclidean metric. Show that $A \cup B$ is a path connected set.
4. (a) Attempt any One of the following: (8)
- (i) Let $\{f_n\}$ be a sequence of real valued functions defined on a set $S \subseteq \mathbb{R}$ such that $f_n \rightarrow f$ uniformly on S . If each f_n is bounded on S , then prove the following:
 - (I) f is bounded on S .
 - (II) There exists $\alpha \in \mathbb{R}^+$ such that $|f_n(x)| \leq \alpha$ for all $n \in \mathbb{N}$ and for all $x \in S$.
 - (ii) State and prove the Cauchy criterion for uniform convergence of a series of functions.
- (b) Attempt any Two of the following: (12)
- (i) Find the pointwise limit of the sequence of functions $f_n : [0, \infty) \rightarrow \mathbb{R}$,

$$f_n(x) = \begin{cases} x & \text{if } x \leq n \\ n & \text{if } x > n \end{cases}$$
 Is the pointwise limit bounded?
 - (ii) Find the radius of convergence and interval of convergence of the following power series:
 - (I) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)\sqrt{n}}$
 - (II) $\sum_{n=0}^{\infty} \frac{(x+3)^{n-1}}{n}$
 - (iii) Show that the series of functions $\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}$ converges uniformly on $[a, \infty)$, $a > 0$.
 - (iv) Consider the power series $\sum_{n=0}^{\infty} c_n x^n$ with integer coefficients. If $c_n \neq 0$ for infinitely many n , then show that its radius of convergence is at most 1.
5. Attempt any Four of the following: (20)
- (a) Prove or disprove: Continuous image of an open ball is an open ball.

- (b) Let $f : [a, b] \rightarrow [a, b]$ be continuous on $[a, b]$ and differentiable on (a, b) . If $\exists c \in \mathbb{R}$ with $0 < c < 1$ such that $|f'(x)| \leq c, \forall x \in (a, b)$ then prove that f is a contraction of $[a, b]$.
- (c) Let (X, d) be a connected metric space. If $f : X \rightarrow (Y, d')$, where d' is a discrete metric, is a continuous function then prove that f is a constant function.
- (d) Prove that there does not exist a continuous onto function from the set $A = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$ to the set $B = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$.
- (e) If $\sum_{n=1}^{\infty} |a_n| < \infty$, then prove that the series $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ converge on \mathbb{R} .
- (f) Let $f_n : [-1, 1] \rightarrow \mathbb{R}, f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$. Given that $f_n \rightarrow f$ uniformly on $[-1, 1]$ where $f(x) = |x|$ for $x \in [-1, 1]$. Find $\lim_{n \rightarrow \infty} \int_{-1}^1 f_n(x) dx$.

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