Duration 3 Hrs | Marks: 100

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. Choose correct alternative in each of the following: (20)

   i. Which of the following maps $d : \mathbb{R} \longrightarrow \mathbb{R}$ is not a metric on $\mathbb{R}$?
   (a) $d(x,y) = |x-y|$
   (b) $d(x,y) = 3|x-y|$
   (c) $d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$
   (d) $d(x,y) = \max\{2, |x-y|\}$

   ii. Which of the following is a description of the open ball $B((0,0),1)$ for the metric $d : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ defined as $d((x_1,x_2),(y_1,y_2)) = |x_1-y_1| + |x_2-y_2|$?
   (a) It consists of the points within the circle of radius 1 centred at $(0,0)$
   (b) It consists of the points inside the square bounded by the lines $x+y=1$, $-x-y=1$, $-x+y=1$, $x-y=1$.
   (c) It consists of the points inside the square bounded by the lines $x=1$, $x=-1$, $y=1$, $y=-1$.
   (d) None of the above

   iii. Let $A = \{(x,y) \in \mathbb{R}^2 : \ x,y \in \mathbb{Q}\}$ and $B = [0,1]\times[0,1]$ be subsets of $\mathbb{R}^2$ with Euclidean metric. Then,
   (a) $\overline{A} = \mathbb{R}^2$; $\overline{B} = B$
   (b) $\overline{A} = \mathbb{Q}\times\mathbb{Q}$; $B^\circ = B$
   (c) $A^\circ = \mathbb{Q}\times\mathbb{Q}$; $B^\circ = (0,1)\times(0,1)$
   (d) None of the above.

   iv. Which of the following sets is not closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ (distance being usual)?
   (a) $\mathbb{Q}$
   (b) $[-\sqrt{3}, \sqrt{3}] \cap \mathbb{Q}$
   (c) $(0,1)\cap\mathbb{Q}$
   (d) $\{2\}$

   v. If $f : [0,1] \to [0,1]$ is defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}\cap[0,1] \\ 1-x & \text{if } x \in (\mathbb{R}\setminus\mathbb{Q})\cap[0,1] \end{cases}$, then
   (a) f is continuous on $[0,1]$ and does not satisfy intermediate value property.
   (b) f satisfies intermediate value property but f is not continuous.
   (c) f is continuous only at $x = 1/2$ and $f([0,1]) = [0,1]$.
   (d) none of the above.

   vi. Which of the following subspaces are dense in $\mathbb{R}$
   (a) $(\mathbb{Z}, d)$, where d is the usual distance.
   (b) $(\mathbb{R}\setminus\mathbb{Q}, d)$, where d is the usual distance.
   (c) $(\mathbb{Q}, d)$, where d is the discrete metric.
   (d) None of these.

   vii. Let (X,d) be a complete metric space , A and B are complete subspaces of (X,d) and $A\cap B$ is nonempty then
   (a) $A\cup B$ and $A\cap B$ are complete.
   (b) $A\cup B$ is complete and $A\cap B$ is not.
   (c) $A\cap B$ is complete and $A\cup B$ is not.
   (d) none of the above.

   viii. Which of the following statements is TRUE?
   (a) $(0,1]$ is compact in $(\mathbb{R}, d)$ where d is usual metric.
   (b) $[1,2]$ is compact in $(\mathbb{R}, d_1)$ where d_1 is the discrete metric.
   (c) $\{1,2,3,4\}$ is a compact set in $(\mathbb{N}, d)$, where d is usual metric from $\mathbb{R}$.
   (d) none of these.

ix. Let A be a compact subset of $\mathbb{R}$. Then
   (a) $\overline{A}$ may not be compact.   (b) A° may not be compact.
   (c) ∂A may not be compact.   (d) None of the above.

x. Let (x_n) be a sequence in $[0,1]$ with usual metric from $\mathbb{R}$. Then, which of the following is **not true**?
   (a) (x_n) has a convergent subsequence.
   (b) (x_n) is bounded but may not be convergent.
   (c) (x_n) may have subsequences converging to different limits.
   (d) (x_n) is Cauchy.

2. (a) Attempt any One from the following: (8)
   (i) Let (X,d) be a metric space and $S \subseteq X$. Show $D(S)$ is a closed subset of X where $D(S)$ denotes the set of all limit points of S.
   (ii) Let (X,d) be a metric space. Prove the following:
      (I) Arbitrary union of open sets is open.
      (II) A subset G of X is open if and only if it is an union of open balls.

   (b) Attempt any Two from the following: (12)
   (i) Show that in a discrete metric space (X,d), every subset is both open and closed.
   (ii) Define a metric space (X,d) and give an example of a metric space. Let (X,d) be a metric space, prove that $|d(x,y) - d(x,z)| \leq d(y,z)\ \forall x,y,z \in X$.
   (iii) Consider the norms $\|\ \|_1$, $\|\ \|_2$ and $\|\ \|_\infty$ on $\mathbb{R}^2$ defined as, for any $x = (x_1,x_2) \in \mathbb{R}^2$, $\|x\|_1 = |x_1| + |x_2|$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. Show that for $x \in \mathbb{R}^2$,
      I) $\|x\|_2 \leq \|x\|_1$ II) $\|x\|_1 \leq \sqrt{2}\,\|x\|_2$ III) $\|x\|_\infty \leq \|x\|_2$
   (iv) Let (X,d) be a metric space. $d_1 : X \times X \to \mathbb{R}$ is a metric defined as $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$, $\forall x,y \in X$. Show that d and d_1 are equivalent metrics on X

3. (a) Attempt any One from the following: (8)
   (i) Let (X,d) be a metric space and Y be a non-empty subset of X. Prove that a subset G of Y is open in the subspace (Y,d) if and only if $G = V \cap Y$ where V is an open set in (X,d) .
   (ii) Show that $[0,1]$ is uncountable.

   (b) Attempt any Two from the following: (12)
   (i) Check whether Cantor's Theorem is applicable in each of the following examples and find $\cap_{n\in\mathbb{N}} F_n$ in each ase, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance d is usual distance from $\mathbb{R}$, in each examples:
      (i) $X = [-1,1]$, $F_n = [-\frac{1}{n}, \frac{1}{n}]$
      (ii) $X = (0,1)$, $F_n = [0, \frac{1}{n}]$
   (ii) Show that in a metric sapce (X,d) every Convergent sequence is Cauchy and the converse not true.

(iii) Let (X, d) be a metric space and $A \subseteq X$. Show that $p \in \overline{A}$ if and only if there is a sequence of points in A converging to p.

(iv) Show that $S = \{x \in \mathbb{Q} : 3 < x^2 < 5\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual metric.

4. (a) Attempt any One from the following: (8)

(i) Consider the metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that if A is closed and bounded then A is sequentially compact.

(ii) Suppose (X, d) is a metric space and $\mathcal{C}$ is a non-empty collection of compact subsets of X then

(I) $\bigcap_{K \in \mathcal{C}} K$ is a compact subset of X.

(II) If $\mathcal{C}$ is finite then $\bigcup_{K \in \mathcal{C}} K$ is a compact subset of X.

(b) Attempt any Two from the following: (12)

(i) Show that a compact subset of a metric space is closed.

(ii) Prove or disprove:

(I) A compact set in a metric space is not open.

(II) Interior of a compact set are compact.

(iii) Consider the metric space $(C[a, b], \| \ \|_\infty)$,where $\|f\|_\infty = \sup\{|f(t)| : t \in [a, b]\}$. Show that the open cover $\{B(0, n)\}_{n \in \mathbb{N}}$ of $C[a, b]$ has no finite subcover. (0 being the constant zero function).

(iv) Prove that a subset K in a discrete metric space (X, d) is compact if and only if K is a finite set.

5. Attempt any Four from the following: (20)

(a) Show that every open ball is an open set.

(b) Let d_1, d_2 be metrics on X. Define $d : X \times X \longrightarrow \mathbb{R}$ as $d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$. Show that d is a metric on X.

(c) Let d_1 and d_2 be metrics on a non-empty set X such that there exist $k_1, k_2 > 0$ such that $k_1 d_1(x, y) \leq d_2(x, y) \leq k_2 d_1(x, y)$, $\forall x, y \in X$. Show that a sequence (x_n) is Cauchy in (X, d_1) if and only if sequence (x_n) is Cauchy in (X, d_2).

(d) Prove that $x^4 - 7x^2 + 10 = 0$ has 4 distinct roots in $\mathbb{R}$.

(e) Determine which of the following subsets of $(\mathbb{R}^2, d)$, where d is Euclidean distance is compact. Justify your answer.

(i) $C = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$

(ii) $D = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}$

(f) If A, B are compact subsets of $\mathbb{R}$ with respect to usual distance, show that $A \times B$ is a compact subset of $\mathbb{R}^2$ with Euclidean metric.

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