(3 Hours) **[Total Marks: 100]**

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose the correct alternative in each of the following: (20)

i. An expression for $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)dydx$ in which the order of integration is reversed is

(a) $\int_{-1}^{1} \int_{-y^2}^{y^2} f(x,y)dxdy.$ (b) $\int_{-1}^{1} \int_{y^2}^{1} f(x,y)dxdy.$

(c) a sum of two integrals (d) None of these

ii. The volume of the solid given by $x^2 + y^2 \leq 1$ and $tan^{-1}\frac{y}{x} \leq z \leq 2\pi$ is

(a) π (b) π^2

(c) 1 (d) None of these

iii. If $f(x,y) = k, k$ constant and $R = [a,b] \times [c,d]$ then $\iint_R kdA$ equals

(a) $k(b-a)(d-c)$ (b) $k(c-a)(d-b)$

(c) $k(b-a)(d-a)$ (d) data insufficient.

iv. The line integral $\int_C \bar{F}\overline{dr}$; $\bar{F} = \frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}$ and $C: x^2 + y^2 = a^2$.

(a) depends on a.
(b) does not exists as Green's Theorem is not applicable.
(c) is a constant independent of a
(d) none of these.

v. The image of [0,1] under the transformation $f: \mathbb{R} \to \mathbb{R}^2$ which is defined as $f(t) = (e^t + e^{-t}, e^t - e^{-t})$, is

(a) An arc of a circle (b) An arc of a parabola

(c) An arc of a hyperbola (d) None of these.

vi. $\int ydx + xdy$ along every closed curve C is

(a) 2π (b) π

(c) $\frac{\pi}{2}$ (d) None of these.

vii. Let $\bar{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$, where P, Q, R are continuously differentiable and S is the surface given by $z = g(x,y), (x,y) \in D$, then $\iint_S F.\hat{n}\,dS$ is given by

(a) $\iint_D \left(P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} + R\right)dxdy$ (b) $\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dxdy$

(c) $\iint_D \left(P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} - R\right)dxdy$ (d) None of these

viii. The surface integral of $F(x,y) = -yi + x\hat{j}$ on S where S is the disc in the XY plane with radius 2 oriented upwards and at the origin is
a) 1. b) –1. c) 0. d) None of these.

ix. The surface integral $\iint_S (\hat{r}\cdot\hat{n})\, dS$ over a closed surface S with volume V is
a) V b) $3V$ c) 0 d) None of these

x. A vector field F is tangent to the boundary of a region S in space. Then $\iiint_S div\ F dV$,
(a) Gauss Theorem is not applicable. (b) 0
(c) depends only on S (d) None of these.

Q.2 a) Attempt **any ONE** question from the following: (08)

i. Define the double integral of a bounded function $f : S \to \mathbb{R}$ where $S = [a,b]\times[c,d]$ is a rectangle in $\mathbb{R}^2$. Further show with usual notations
$m(b-a)(d-c) \le \iint_S f \le M(b-a)(d-c)$.

ii. Prove that a continuous function is integrable for a rectangular domain in $\mathbb{R}^2$.

b) Attempt **any TWO** questions from the following: (12)

i. State the change of variable formula for triple integral clearly stating the conditions under which it is valid. Express further, how will you use to express the triple integral in spherical coordinates.

ii. Evaluate the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-y^2}\, dydx$ by reversing the order of integration.

iii. Using a suitable change of variable, evaluate $\iint_S (x^2+y^2)\, dxdy$ where S is the region in the XY-plane bounded by the curves
$x^2-y^2=1,\ x^2-y^2=2, xy=2, xy=4$.

iv. Find the mass for a plate bounded by $x=0, x=1, y=0, y=2$ whose density is $\delta(x,y)=$ x. [hint: mass = $\iint \delta(\mathrm{x},\mathrm{y})\, dA$]

Q.3 a) Attempt **any ONE** question from the following: (08)

i. If $F=(P,Q)$ is continuously differentiable function defined on a simply connected region D in $\mathbb{R}^2$, then show that $\oint Pdx + Qdy = 0$ around every closed curve C in D if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\forall (x,y) \in D$.

ii. State and prove Green's theorem for a rectangular region.

b) Attempt **any TWO** questions from the following: (12)

i. If f is continuously differentiable scalar field defined on an open set U in $\mathbb{R}^n$ and C is a simple, smooth, closed curve in U with parameterization $r(t), t \in [a,b]$, then prove that $\int_C \nabla f . dr = 0$.

ii. Using Green's Theorem, find the area of the region D which is bounded by lines $y = 1, y = 3, x = 0$ and the parabola $y^2 = x$.

iii. Verify Green's Theorem for the function $F(x, y) = (2xy - x^2, x + y^2)$ over the region D bounded by positively oriented curve C formed by parabolas $y = x^2$ and $x = y^2$.

iv. Find whether the force field $F(x, y, z) = (ysinz, xsinz, xycosz)$ is conservative. If so find ϕ so that $F = \nabla\phi$ and calculate the work done in the moving the particle form the point $P(0,0,0)$ to the point $Q(\pi, \pi, \pi)$.

Q.4 a) Attempt **any ONE** question from the following: (08)

i. State and prove the Stokes' Theorem for an oriented smooth, simple parameterized surface in $\mathbb{R}^3$ bounded by a simple, closed curve traversed counter clockwise assuming general form of Green's Theorem.

ii. Let $S = r(T)$ be a smooth parametric surface described by a differentiable function r defined on region T. Let f be a scalar field defined and bounded on S. Define surface integral of f over S. If R and r are smoothly equivalent functions, $R(s,t) = r(G(s,t))$ where $G(s,t) = u(s,t)\hat{\imath} + v(s,t)\hat{\jmath}$ being continuously differentiable. Then show that $R_s \times R_t = \frac{\partial(u,v)}{\partial(s,t)} (r_u \times r_v)$. Further prove that $\iint_{r(A)} f dS = \iint_{R(B)} f d S$ where $G(B) = A$.

b) Attempt **any TWO** questions from the following: (12)

i. Find the surface area of S where S is the part of the plane $x + 2y + z = 4$ that lies inside the cylinder $x^2 + y^2 = 4$.

ii. Evaluate the surface integral of vector field F over S where $F(x, y, z) = (x, y, z)$ and S is the paraboloid $z = x^2 + y^2 - 1, -1 \leq z \leq 0$ oriented upwards

iii. Prove the following identities, assuming S and V satisfy the conditions of the Divergence Theorem and scalar fields f and g, components of $\bar{F}$ have continuous second order partial derivatives, $\hat{n}$ is unit outward normal to S.

a) $\iint_S (f\nabla g) \cdot \hat{n} dS = \iiint_V (f\nabla^2 g + \nabla f \cdot \nabla g) dV \cdot$

b) $\iint_S (f\nabla g - g\nabla f) \cdot \hat{n} dS = \iiint_V (f\nabla^2 g - g\nabla^2 f) dV.$

iv. Evaluate the surface integral of the vector field $F(x, y, z) = (z, y, x)$ over the unit sphere $x^2 + y^2 + z^2 = 1$ using the Gauss divergence theorem.

Q.5 Attempt **any FOUR** questions from the following: (20)

a) Evaluate $\iiint_S dV$ where region S is bounded by the three co-ordinate planes and the plane $2x + y + 3z = 1$.

b) Find area of the region S which is bounded by the parabola $x = 9 - y^2$ and $x = 0$.

c) Evaluate the line integral of the vector field $F(x, y, z) = (xz, y + z, x)$ along the curve : $x(t) = e^t, y(t) = e^{-t},\ z(t) = e^{2t}, 0 \leq t \leq 1$.

d) Evaluate the integral of the scalar field $f(x, y) = \sin x + \cos y$ along the line segment from (0,0) to $(\pi, 2\pi)$.

e) Evaluate surface integral of a scalar field f over S where $f(x, y, z) = x$ and S is the part of the plane $x + 2y + 3z = 6$ in the first octant.

f) Using Stokes' Theorem evaluate the line integral $\oint_C \overline{F} \cdot d\,\overline{r}$ where $F(x, y, z) = (4y, 2z, 6y)$ and C is the curve of intersection of $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$.
