(3 Hours) **[Total Marks: 100]**

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose correct alternative in each of the following: (20)

i. The integral $\int_0^2 \int_x^{x\sqrt{3}} f(\sqrt{x^2+y^2})\, dydx$ in polar coordinates is

(a) $\int_0^{\pi/4} \int_0^{2sec\theta} f(r)rdrd\theta$ (b) $\int_0^{\pi/3} \int_0^{2sec\theta} f(r)rdrd\theta$

(c) $\int_{\pi/4}^{\pi/3} \int_0^{2sec\theta} f(r)rdrd\theta$ (d) $\int_{\pi/4}^{\pi/3} \int_0^{2cos\theta} f(r)rdrd\theta$

ii. $I = \int_0^1 \int_{x^2}^{x} xf(y)dydx$ where f is continuous function defined on [0,1]. Then I is

(a) $\frac{1}{2}\int_0^1 (y-y^2)f(y)dy$ (b) Independent of $f(y)$

(c) $\frac{1}{2}\int_0^1 (y^2-y)f(y)dy$ (d) $f(x)$

iii. The volume of region bounded by $z = x+y, z = 6, x = 0, y = 0, z = 0$ is

(a) 36 cubic units (b) 30 cubic units

(c) 216 cubic units (d) None of these.

iv. The parametric equations $x = \cos(\cos t)\, , y = \sin(\cos t)\, , t \in [\, 0\, , \pi]$ describes

(a) one full circle
(b) an arc of a circle in first quadrant
(c) an arc of a circle in the first and fourth quadrant
(d) None of the above.

v The line integral $\int_c \bar{F}\cdot\overline{dr}$; $\bar{F} = \frac{-y\,\hat{\imath}+x\hat{\jmath}}{x^2+y^2}$ and $C: x^2+y^2 = a^2$.

(a) depends on a
(b) does not exist as Green's Theorem is not applicable
(c) is a constant independent of 'a'
(d) None of the above

vi. $\nabla f(x,y,z) = 2xyze^{x^2}\hat{\imath} + ze^{x^2}\hat{\jmath} + ye^{x^2}\hat{k}$ and $f(0,0,0) = 7$. Then $f(1,1,2)$ =?

(a) $2e + 5$ (b) $2e + 7$

(c) $e + 5$ (d) $7e + 2$

vii. The magnitude of the fundamental vector product $\frac{\partial\bar{r}}{\partial u}\times\frac{\partial\bar{r}}{\partial v}$ for surface $\bar{r}(u,v) = (u+v)\hat{\imath} + (u-v)i + 4k$ is

(a) $\sqrt{4+v^2}$ (b) $\sqrt{4+128}v^2$

(c) $\sqrt{4v^2+1}$ (d) None of these

viii. The flux of the vector field $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$ equals

(a) $\frac{4}{3}\pi$ (b) $\frac{2}{3}\pi$

(c) $\frac{1}{3}\pi$ (d) None of these

ix. The surface integral $\iint_S (\bar{r} \cdot \hat{n})dS$ over a closed surface S with volume V, where $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ is

(a) V (b) $3V$

(c) 0 (d) None of these

x. $div\big(curl(x^2, yz, \sin z)\big)$ is

(a) $2x + z + \cos z$ (b) 0

(c) $\bar{0}$ (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Define the double integral of a bounded function $f: S \to \mathbb{R}$ where $S = [a, b] \times [c, d]$ is a rectangular region in $\mathbb{R}^2$ and using usual notation show that $m(b-a)(d-c) \le \iint_S f \le M(b-a)(d-c)$.

ii. State and prove Fubini's Theorem for a rectangular domain in $\mathbb{R}^2$.

b) Attempt any TWO questions from the following: (12)

i. Prove that every continuous function defined on a rectangular domain D in $\mathbb{R}^2$ is integrable.

ii. Evaluate the integral $\iint_S 3y\,dA$ using polar coordinates, where S is the region in the first quadrant bounded above by the circle $(x-1)^2 + y^2 = 1$ and below by the line $y = x$.

iii. Using cylindrical co-ordinates find the volume of the solid region S in $\mathbb{R}^3$ which is bounded by the paraboloid $x^2 + y^2 = 4 - z$ and the plane $z = 0$.

iv. Evaluate $\iiint_S \frac{dxdydz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ where S is region in $\mathbb{R}^3$ between the two spheres with centre at the orgin and radii 2 and 5.

Q.3 a) Attempt any ONE question from the following: (08)

i. Let f be a continuously differentiable scalar field defined on an open set U in $\mathbb{R}^n$. Suppose P, Q are two points of U that can be connected by piecewise smooth curve C lying in U. Prove that $\int_C \nabla f \cdot dr = f(Q) - f(P)$ given that C has parameterization $r(t)$, $t \in [a, b]$ with $r(a) = P$ and $r(b) = Q$. Further if $F = \nabla f$ where $(x, y) = cos(x + 2y)$, does there exist a smooth, closed path C such that $\int_C F.dr = \pi$? If so, find such a path C.

ii. State and prove Green's Theorem for a rectangle. Evaluate $\oint_c \left(3y - e^{\cos x)}\right)dx + (7x + \sqrt{y^5 + 1}\,)\,dy$ where C is the circle $x^2 + y^2 = 9$.

b) Attempt any TWO (12)

i. Show that the vector $F = (y \sin z,\ x sin\, z, xy \cos z)$ is conservative. If so find f such that $= \nabla f$.

ii. Using Green's Theorem, find the area of the region D whose boundary is positively oriented simple closed curve bounded by the lines $y = 1, y = 3, x = 0$ and the parabola $y^2 = x$.

iii. Show that two equivalent parameterized curves in $\mathbb{R}^n$ have essentially the same image set. Show that the converse is not true by considering curves $\alpha(t)$= (cost, sint) and $\beta(t) =$ (sint, cost), $0 \le t \le 2\pi$.

iv. $F = (P, Q)$ is a continuously differentiable function defined on a simply connected region D in $\mathbb{R}^2$. Show that $\int_c Pdx + Qdy = 0$ around every piecewise smooth closed curve C in D if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\ \forall\,(x, y) \in D.$

Q.4 a) Attempt any ONE. (08)

i. Let $S = \bar{r}(T)$ be a smooth parametric surface described by a differentiable function $\bar{r}$ defined on region T. Let f be defined and bounded on S. Define surface integral of f over S. If R and r are smoothly equivalent functions, $R(s,t) = \bar{r}(G(s,t))$ where $G(s,t) = u(s,t)\hat{\imath} + v(s,t)\hat{\jmath}$ being continuously differentiable. Then show that $\iint_{r(A)} f dS = \iint_{R(B)} f d\,S$ where $G(B) = A$.

ii. State Divergence Theorem for a solid in 3-space (or $\mathbb{R}^3$) bounded by an orientable closed surface with positive orientation and prove the divergence Theorem for cubical region.

b) Attempt any TWO . (12)

i. Let $S = \bar{r}(T)$ be a smooth parametric surface in uv plane. Define area of S. If S is represented by an equation $z = f(x,y)$ then show that area of S is given by $\iint_T \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ \ dxdy\ \ where\ T$ is projection of S on XY −plane.

ii. Evaluate the surface integral of $F(x,y,z) = (x,y,0)$ over S, where S is the hemisphere above XY-plane of radius 2.

iii. Find surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

iv. Use Stokes' theorem to evaluate $\iint_S (curl\,F) \cdot ndS$ where $F(x,y,z) = y\hat{\imath} + x\hat{\jmath} + xz\hat{k}$ where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1;\ z \geq 0$ and n is the unit normal with a non-negative z component.

Q.5 Attempt any FOUR. (20)

a) Using double integration, find the area of the region S in $\mathbb{R}^2$ bounded by the parabola $y = 9 - x^2$ and $y = x^2 + 1$.

b) Evaluate $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

c) Let U be an open set in $\mathbb{R}^n$ and $\alpha : [a,b] \to U$ be a parameterization of curve Γ. If $f, g : U \to \mathbb{R}$ are continuous functions, then prove that

$$\int_\Gamma (cf + dg) = c\int_\Gamma f + d\int_\Gamma g$$, where c, d are real constants.

d) Evaluate the integral of the vector field, $F(x,y,z) = (x,^2 - xy, 1)$ along the circle of radius 1, with centre at the origin and lying in the yz, plane, traversed counterclockwise as viewed from the positive x axis.

e) Use Gauss Divergence theorem to evaluate $\iint_S F \cdot ndS$ where $F(x,y,z) = (x + y,\ y + z, z + x)$ and S is the region given by $-4 \leq z \leq 4;\ 0 \leq x^2 + y^2 \leq 4$.

f) Evaluate the surface integral of $f(x,y,z) = z$ over the surface parameterized by $\alpha\,(u,v) = (u\cos v, u\sin v, 1); 1 \leq u \leq 2, 0 \leq v \leq 2\pi$

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