3 Hours] **[Total Marks: 100**

N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

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1. Fill in the blank by choosing the correct option.

(i) Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ and $V = \{B \in M_4(\mathbb{R}) : AB = BA\}$. Then, (2)

(a) $\dim V = 4$ and $\dim M_4(\mathbb{R})/V = 12$
(b) $\dim V = 8$ and $\dim M_4(\mathbb{R})/V = 8$
(c) $\dim V = 8$ and $\dim M_4(\mathbb{R})/V = 16$
(d) None of these.

(ii) Let α be an orthogonal transformation of the plane such that the matrix of α w. r. t. the standard basis of $\mathbb{R}^2$ is $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, then α represents a (2)

(a) rotation about origin through $\frac{\pi}{4}$.
(b) rotation about origin through $\frac{5\pi}{4}$.
(c) rotation about the line $y = -x$. (d) None of the above.

(iii) A 2×2 matrix A has the characteristic polynomial $x^2 + 2x - 1$, then the value of det $(2I_2 + A)$ is (2)

(a) $\dfrac{1}{det\ A}$ (b) 0 (c) $2 + det\ A$ (d) $2\,det\ A$

(iv) Let A and B be square matrices such that $AB = I$ then zero is an eigenvalue of (2)

(a) A but not B (b) B but not A
(c) both A and B (d) neither A nor B.

(v) If λ is a characteristic root of a matrix A then characteristic roots of $-A$ and $\alpha I - A$ respectively are (2)

(a) $-\lambda$ and $\alpha - \lambda$ (b) $-\lambda$ and α
(c) $-\lambda$ and λ (d) None of these.

(vi) Which of the following statements are true (2)

(p) If the characteristic roots of two $n \times n$ matrices are same then their characteristic polynomials are same.

(q) If the characteristic polynomials of two $n \times n$ matrices are same then their characteristic roots are same.

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(r) If eigen values of two $n \times n$ matrices are same then their eigen vectors are same.

(s) The characteristic roots of two $n \times n$ matrices are same but their characteristic polynomials may not be same.

(a) (q) and (s) are true. (b) (p), (r) are true.
(c) (p), (q) and (r) are true. (d) only (q) is true.

(vii) The minimal polynomial of the diagonal matrix $A =$ diag $\{1, -1, 1, -1\}$ is (2)
(a) $x^2 + 1$ (b) $x^2 - 1$ (c) $(x^2 - 1)^2$ (d) None of these.

(viii) Let $A = \begin{bmatrix} 0 & a \\ 0 & -a \end{bmatrix}$ (2)
(a) A is orthogonally diagonalizable if and only if $a = 1$
(b) A is not diagonalizable for any $a \in \mathbb{R}$.
(c) A is diagonalizable but not orthogonally diagonalizable.
(d) None of these.

(ix) If $A, B, C, D \in M_2(\mathbb{R})$ such that A, B, C, D are non-zero and not diagonal. If $A^2 = I, B^2 = B, C^2 = 0, C \neq 0$ and every eigenvalue of D is 2, then (2)
(a) A, B, C, D are all diagonalizable.
(b) B, C, D are diagonalizable.
(c) A, B are diagonalizable
(d) Only D is diagonalizable.

(x) The quadratic form $Q(x) = x_1^2 + 4x_1x_2 + x_2^2$ has (2)
(a) rank = 1, signature = 1. (b) rank = 2, signature = 0.
(c) rank = 2, signature = 2. (d) None of the above.

2. (a) Answer any **ONE**

(i) Let V be a finite dimensional inner product vector space and $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent. (8)
(p) T is orthogonal.
(q) $\|T(X)\| = \|X\|$ for all $X \in V$.
(r) If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V, then $\{T(e_i)\}_{i=1}^n$ is also an orthonormal basis of V.

(ii) State and prove the Cayley Hamilton Theorem. (8)

(b) Answer any **TWO**

(i) State and prove the 'First Isomorphism Theorem of vector space' (Fundamental theorem of vector space homomorphism). (6)

(ii) Let $(V, <>)$ be an n dimensional inner product space and W be a subspace of V of dimension $n - 1$. Let u be a unit (6)

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vector orthogonal to W. Show that $T : V \to V$ defined by $T(x) = x - 2\langle x, u\rangle u$ is an orthogonal linear transformation such that $T(w) = w,\ \forall\, w \in W$ and $T(u) = -u$.

(iii) Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \end{pmatrix}$. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x) = AX$(X being a column vector in $\mathbb{R}^3$). Find $kerT$,a basis of $kerT$ and $\mathbb{R}^3/kerT$. (6)

(iv) Show that $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\alpha(x, y, z) = (\frac{x}{2} + \frac{\sqrt{3}}{2}z - 1, y, \frac{\sqrt{3}}{2}x - \frac{z}{2} + 5)$ is an isometry. Express it as a composite of an orthogonal transformation and a translation. (6)

3. (a) Answer any **ONE**

(i) Define eigen value of a real square matrix. Show that, if λ is an eigen value of a real $n \times n$ matrix A, then (8)

(p) λ is an eigen value of A^t.

(q) λ^k is an eigen value of A^k for $k \in \mathbb{N}$. Hence $f(\lambda)$ is an eigen value of $f(A)$, for a polynomial $f(x)$ over $\mathbb{R}$.

(r) If A is invertible, then λ^{-1} is an eigen value of A^{-1}.

(ii) Show that minimal polynomial of a real matrix $A_{n\times n}$ divides every polynomial which annihilates A. Further show that λ is a root of the minimal polynomial of matrix A if and only if λ is a characteristic root of A. (8)

(b) Answer any **TWO**

(i) If A is an $n\times n$ real matrix, and $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigen value of A with $X_1, X_2, \cdots, X_k$ as corresponding eigenvectors, then show that $X_1, X_2, \cdots, X_k$ are linearly independent. (6)

(ii) Let $A_{n\times n}$ be a real matrix. if A has n distinct characteristic roots, then prove that the characteristic polynomial of A = the minimal polynomial of A. (6)

(iii) Find the eigen values and eigen vectors of $A_{2\times 2}$ such that (6)

$$\sum_{j=1}^{2} a_{ij} = 1 \text{ for } i = 1, 2.$$

(iv) Let v be a non-zero vector in $\mathbb{R}^n$ and $A = vv^T$ where v is treated as a $n\times 1$ column vector. Find the minimal polynomial of A. (6)

4. (a) Answer any **ONE**

(i) Define an orthogonally diagonalizable matrix. Show that every real symmetric matrix is orthogonally diagonalizable. (8)

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(ii) Let A be real symmetric matrix of order n. Show that characteristic roots of A are real. Also show that if λ_1, λ_2 are distinct eigen values of A and X_1, X_2 are corresponding eigen vectors then X_1, X_2 are orthogonal. (8)

(b) Answer any **TWO**

(i) Show that an $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a basis consisting of eigen vectors of A. (6)

(ii) Show that every quadratic form $Q(x_1, x_2, \cdots x_n)$ over $\mathbb{R}$ can be reduced to standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by an orthogonal change of variables $X = PY$,where X, Y are column vectors in $\mathbb{R}$ and P is an $n \times n$ orthogonal matrix. (6)

(iii) For $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix. Hence or otherwise find A^{1000} and B such that $B^3 = A$. (6)

(iv) Let A be a 13×13 real matrix of rank 1. Find the eigen values and show that geometric multiplicity is equal to algebraic multiplicity for each eigen value. (6)

5. Answer any **FOUR**

(a) Find the orthogonal transformations in $\mathbb{R}^3$ which represent reflections with respect to $2x - y + z = 0$. (5)

(b) Express the characteristic polynomial of $aI + bA$ interms of the characteristic polynomial of A. (5)

(c) Prove or disprove:Two matrices are similar if and only if their characteristic polynomials are same. (5)

(d) Define invariant subspace with respect to a linear transformation. Check which of $V = \{(x, 0) : x \in \mathbb{R}\}$ and $W = \{(x, -x) : x \in \mathbb{R}\}$ are $T : \mathbb{R}^2 \to \mathbb{R}^2$ invariant where $T(x, y) = (y, x)$. Also identify the eigen value of T with respect to which W is an eigen space. (5)

(e) If $A^2 = A$ for non-zero $n \times n$ matrix A then show that algebraic multiplicity of eigen value 1 is rank A. (5)

(f) Find value of k, for which the symmetric matrix associated to the quadratic form $3x_1^2 + x_2^2 + 2x^2 + 2x_1x_3 + 2kx_2x_3$ is positive definite. State the result used. (5)

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